MATHEMATICAL ANALYSIS

FIRST EDITION
2023

Topics :-

- ★ **Function of Bounded Variation and Rectifiable Curves**
- ★ **The Riemann-Stieltjes Integral**

A Book for Graduate & Post Graduate Students

- Ashwani Goyal

MATHEMATICAL ANALYSIS

First Edition : 2023

For

Graduate and Post Graduate Students

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To My Mother

PREFACE

The present book is a culmination of very honest and sincere efforts, keeping in mind the level and aspirations of Mathematics Students at the post graduation level.

I have minutely scanned the whole manuscript of **Mathematical Analysis** and edited, wherever necessary. Many observations out of my own long year of classroom teaching experience regarding the difficulties of students have been kept in my mind while presenting and editing the subject matter.

I have not changed Mathematics (nobody can), but I have certainly changed the style of its presentation to make things simpler and easily understandable to our students, for whom the book has been written. Surely, my esteemed friends while patronizing this book for their classroom instruction, will also enjoy it. I honestly feel everyone will find the book refreshingly different and acceptable.

A serious effort has been made to keep the book free from errors, but even then some errors might exist. Suggestions for the improvement of the book will be gratefully acknowledged.

I am much obliged to Prof. Dr. Parminder Singh, Principal, Govt. (State) College of Education Patiala, who helped me all along with valuable suggestions.

My sincere thanks to my friend Prof. Ashwani Kumar & Dr. Sukhwinder Singh Government Ranbir College, Sangrur, who helped me in various ways; and , in particular, to Dolly Goyal and Pawan Kumar who went through the first draft of the manuscript and suggested a number of improvements in the presentation of the subject.

Last but not least, I would like to express my gratitude to Mr. Amit Arora Insp. Food Supply at Sangrur.

Ashwani Goyal

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Chapter -1

Monotone Functions

1.1. Definition: Let $f:[a,b]\to R$ be a function. Then f is said to be

Increasing on $[a,b]$ if for every $x,y\in[a,b]$ $x<y \Rightarrow f(x)\le f(y)$

Decreasing on $[a,b]$ if for every $x,y\in[a,b]$ $x<y \Rightarrow f(x)\ge f(y)$

Monotone if f is either increasing or decreasing on $[a,b]$

If the interval $[a,b]$ can be divided into a finite number of intervals such that f is monotone on each of them then f is said to be piecewise monotone on $[a,b]$.

Bounded and unbounded sets : Supremum, Infimum

A subset S of real numbers is said to be **bounded above** if $\exists$ a real number K such that every number of S is less than or equal to k, i.e. $x\le k\ \forall\ x\in S$

The number k is called an **upper bound** of S. If no such number k exists, the set is said to be **unbounded above or not bounded above.**

The set S is said to be **bounded below** if $\exists$ a real number k such that every number of S is greater than or equal to k, i.e. $x\ge k\ \forall\ x\in S$

The number k is called an **lower bound** of S. If no such number k exists, the set is said to be **unbounded below or not bounded below.**

A set S is said to be **bounded** if it is bounded above as well as below.

If the set of all upper bound of a set S has the smallest member say α, then α is called the **supremum or least upper bound** of S. The fact that **supremum** α is the smallest of all upper bounds of S may be described by the following two properties:

(i) $x\le\alpha$, $\forall x\in S$

(ii) for any positive number ε, however small, $\exists$ a number $y\in S$ such that $y>\alpha-\varepsilon$

If the set of all lower bound of a set S has the greatest member say β, then β is called the **infimum or greatest lower bound** of S. The fact that **infimum** β is the greatest of all lower bounds of S may be described by the following two properties:

(i) $x \geq \beta$, $\forall x \in S$

(ii) for any positive number ε, however small, $\exists$ a number $y \in S$ such that $y < \beta + \varepsilon$

1.2 Properties of Monotonic Functions

Theorem 1.2.1. Let f be monotonically increasing on $[a,b]$. Then if c is any point such that $a < c < b$, $f(c-0)$ and $f(c+0)$ both exist and

$$f(c-0) = \sup_{a<x<c} f(x), \quad f(c+0) = \inf_{c<x<b} f(x)$$

Also $f(c-0) \leq f(c) \leq f(c+0)$

Proof: The set $\{f(x) : a < x < c\}$ is bounded above. In fact f(c) is an upper bound of this set. Let the supremum of this set be denoted by α. We have $\alpha \leq f(c)$.

Let $\varepsilon > 0$ be given. We have $f(x) \leq \alpha < \alpha + \varepsilon \ \forall \ x \in (a,c)$

Also there exists $\delta > 0$ such that $a < c - \delta < c$

and $f(c-\delta) > \alpha - \varepsilon$ { by properties of supremum}

Also because f is monotonically increasing,

$$f(x) > f(c-\delta) > \alpha - \varepsilon \text{ when } x > c - \delta$$

Thus we see that there exists $\delta > 0$ such that $\alpha - \varepsilon < f(x) < \alpha + \varepsilon$

When $c - \delta < x < c$

$$\Rightarrow f(c-0) = \lim_{x \to (c-0)} f(x) = \alpha = \sup_{a<x<c} f(x)$$

Similarly prove that $f(c+0) = \inf_{c<x<b} f(x)$

Corollary 1.2.1 Let f be monotonically increasing in $[a,b]$. Then $f(a+0), f(b-0)$ exist and $f(a+0) = \inf_{a<x<b} f(x)$ and $f(b-0) = \sup_{a<x<b} f(x)$

Corollary 1.2.2 Let f be monotonically decreasing in $[a,b]$. Then if c is any point such that $a<c<b$, $f(c-0)$ and $f(c+0)$ both exist and

$$f(c-0) = \inf_{a<x<c} f(x),\ f(c+0) = \sup_{c<x<b} f(x),$$

Also $f(c-0) \geq f(c) \geq f(c+0)$

Corollary 1.2.3 Let f be monotonically decreasing in $[a,b]$. Then $f(a+0), f(b-0)$ exist and $f(a+0) = \sup_{a<x<b} f(x)$ and $f(b-0) = \inf_{a<x<b} f(x)$

Theorem 1.2.2 Let f be an increasing function defined on $[a,b]$ and let $x_0 < x_1 < x_2 < \dots\dots < x_n$ be n+1 points such that $a = x_0 < x_1 < x_2 < \dots\dots < x_n = b$. Then we have the inequality

$$\sum_{k=1}^{n-1} \left[f\left(x_k^{+}\right) - f\left(x_k^{-}\right) \right] \leq f(b) - f(a)$$

Proof: Let $y_k \in (x_k, x_{k+1})$

For $1 \leq k \leq n-1$, we have

$f\left(x_k^{+}\right) \leq f(y_k)$ and $f\left(x_k^{-}\right) \geq f(y_{k-1})$

$\therefore\ f\left(x_k^{+}\right) - f\left(x_k^{-}\right) \le f(y_k) - f(y_{k-1})$

For k = 1, 2, ……..,,(n-1) we have

$f\left(x_1^{+}\right) - f\left(x_1^{-}\right) \le f(y_1) - f(y_0)$,

$f\left(x_2^{+}\right) - f\left(x_2^{-}\right) \le f(y_2) - f(y_1)$

…………………………………………………….

$f\left(x_{n-1}^{+}\right) - f\left(x_{n-1}^{-}\right) \le f(y_{n-1}) - f(y_{n-2})$

Adding above, we get

$$\sum_{k=1}^{n-1}\left[f\left(x_k^{+}\right) - f\left(x_k^{-}\right)\right] \le f(y_{n-1}) - f(y_0)$$

As $f(y_{n-1}) - f(y_0) \le f(b) - f(a)$, we get

$$\sum_{k=1}^{n-1}\left[f\left(x_k^{+}\right) - f\left(x_k^{-}\right)\right] \le f(b) - f(a)$$

Theorem 1.2.3 Let f be monotone on $[a,b]$, then the set of points of $[a,b]$ at which f is discontinuous is almost countable.

Proof: Suppose for the sake of definiteness f is monotonic increasing and let E be the set of points at which f is discontinuous. With every point x of E we associate a rational number r(x) of E such that $f\left(x^{-}\right) < r(x) < f\left(x^{+}\right)$

Let $x_1 \ne x_2$

Take $x_1 < x_2 \Rightarrow f\left(x_1^{+}\right) \le f\left(x_2^{-}\right)$ …………..(i)

$\because\ f\left(x_1^{-}\right) < r(x_1) < f\left(x_1^{+}\right)$ …………..(ii)

$f\left(x_2^{-}\right) < r(x_2) < f\left(x_2^{+}\right)$ ……………(iii)

From (i), (ii) and (iii) we have $r(x_1) < r(x_2)$

Thus we have establish a one -one corresponding between the set E and a subset of set of rational numbers. The latter set as we know is countable. Hence E is at most countable.

Chapter -2

Function of Bounded Variation and Rectifiable Curves

2.1. Definition Let f be defined on [a, b].

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ is a partition of [a. b].

Define $V(f, a, b)$ or $V_a^b(f)$ = $\sup\limits_{P \in P[a,b]} \sum\limits_{i=1}^{n} |f(x_i) - f(x_{i-1})|$, then $V_a^b(f)$ is said to be total variation of f over [a, b]. It is also affiliated as $V(f)$ or V_f

If $V_a^b(f)$ is finite or $V_a^b(f) < +\infty$, then f is said to be function of bounded variation.

Note. $V_a^b(f) = 0$ ifand only if f is constant on $[a, b]$

Theorem 2.1.1. If f is bounded monotonic on [a. b] then f is bounded variation on [a, b]

Proof. Let f be monotonic increasing on [a, b] and
$P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ is a partition of [a. b].

Then $V_a^b(f)$= $\sup\limits_{P \in P[a,b]} \sum\limits_{i=1}^{n} |f(x_i) - f(x_{i-1})|$,

$$= \sup_{P \in P[a,b]} \left\{ \begin{matrix} |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + |f(x_3) - f(x_2)| + \\ \ldots\ldots\ldots\ldots + |f(x_n) - f(x_{n-1})| \end{matrix} \right\}$$

$x_i > x_{i-1} \Rightarrow f(x_i) \geq f(x_{i-1})$ Where $i = 1,2,3,\ldots.,n$

$$= \sup_{P \in P[a,b]} \{f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots\ldots + f(x_n) - f(x_{n-1})\}$$

$$= \sup_{P \in P[a,b]} \{f(b) - f(a)\} = f(b) - f(a) \text{ = finite}$$

$\therefore$ f is of function of bounded variation on $[a,b]$ and $V_a^b(f)$ = $f(b) - f(a)$

Similarly a monotonically decreasing bounded function is of bounded variation with total variation $= f(a) - f(b)$.

Thus for a bounded monotonic function f

$$V_a^b(f) = |f(b) - f(a)|$$

Example 2.1.1. Show that sin x is bounded variation over a $\left[0, \frac{\pi}{2}\right]$

Sol. Clearly f(x) = sin x is increasing in $\left[0, \frac{\pi}{2}\right]$. So f is function of bounded variation.

Theorem 2.1.2. If f is continuous on [a, b] and f' exists and is bounded on (a,b) then the function f is of bounded variation on [a, b]

Proof. Since f' is bounded on [a, b]

$$\therefore |f'(x)| \leq k \ \forall \ x \in [a,b]$$

Let $P=\{x_0, x_1, x_2, \ldots\ldots, x_n\}$ is a partition of [a. b].

$$V_a^b(f) = \sup_{P\in P[a,b]} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

By Mean value theorem

$$= \sup_{P\in P[a,b]} \sum_{i=1}^{n} |f'(t)||x_i - x_{i-1}| \le k \sup_{P\in P[a,b]} \sum_{i=1}^{n} (x_i - x_{i-1}) \; (\because x_i > x_{i-1})$$

$= k(b-a)$ = finite

$\therefore$ f is function of bounded variation.

Example 2.1.2. $f(x)=\begin{cases} x^2\cos\dfrac{1}{x} & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$ Show that f(x) is bounded variation on [0, 1]

Sol. It is clear that f is continuous on $[0,1]$

$$f'(x)=\begin{cases} \sin\dfrac{1}{x}+2x\cos\dfrac{1}{x} & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}$$

$\therefore$ $f'(x)$ exist in (0,1)

$$|f'(x)| = \left|\sin\frac{1}{x}+2x\cos\frac{1}{x}\right| \le \left|\sin\frac{1}{x}\right| + 2|x|\left|\cos\frac{1}{x}\right| \le 1+2=3 \; (\because |x|\le 1)$$

$\therefore$ f is function of bounded variation.

Note: Bounded ness of f' is a sufficient condition it is not necessary condition.

Example 2.1.3 Let $f:[0,8]\to R$ defined by $f(x)=x^{1/3}$

Clearly f(x) is increasing

$\therefore$ f is function of bounded variation.

But $f'(x)=\dfrac{1}{3x^{2/3}}\to\infty$ as $x\to 0$

Note: A continuous function may not a function of bounded variation.

Example2.1.4. Give example to show that a continuous function may not be function of bounded variation.

Sol. Let $f(x)=\begin{cases} x\cos\left(\dfrac{\pi}{2x}\right) & , \quad 0<x\le 1 \\ 0 & , \quad x=0 \end{cases}$

First we prove f is continuous on [0.1]

In $0<x\le 1$, clearly f is continuous

At x = 0, f(0) = 0 , by definition for $|x-0|<\delta$

$$|f(x)-f(0)|=\left|x\cos\left(\frac{\pi}{2x}\right)-0\right|=|x|\left|\cos\left(\frac{\pi}{2x}\right)\right|=|x|<\delta=\varepsilon$$

$\therefore$ f is continuous on [0.1]

Next we prove f is not function of bounded variation.

Let a partition of [0, 1] such that $P=\left\{0,\dfrac{1}{2n},\dfrac{1}{2n-1},\dfrac{1}{2n-2}.....,\dfrac{1}{3},\dfrac{1}{2},1\right\}$

By definition $V_0^1(f) = \sup\limits_{P \in P[0,1]} \sum\limits_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$

$$= \sup_{P \in P[0,1]} \left\{ \begin{array}{l} \left| f(x_1) - f(x_0) \right| + \left| f(x_2) - f(x_1) \right| + \left| f(x_3) - f(x_2) \right| + \\ \text{............} + \left| f(x_n) - f(x_{n-1}) \right| \end{array} \right\}$$

$f(x_0) = f(0) = 0$, $f(x_1) = f\left(\frac{1}{2n}\right) = \frac{1}{2n}\cos(n\pi) = \frac{(-1)^n}{2n}$,

$$f(x_2) = f\left(\frac{1}{2n-1}\right) = \frac{1}{2n-1}\cos\frac{(2n-1)\pi}{2}$$

$$= \frac{1}{2n-1}\cos\left(n\pi - \frac{\pi}{2}\right) = \frac{1}{2n-1}\sin n\pi = 0$$

$$f(x_3) = f\left(\frac{1}{2n-2}\right) = \frac{1}{2n-2}\cos\frac{(2n-2)\pi}{2}$$

$$= \frac{1}{2n-2}\cos(n\pi - \pi) = -\frac{1}{2n-2}\cos n\pi = \frac{(-1)^{n+1}}{2n-2}$$

....

$f(x_{n-2}) = f\left(\frac{1}{3}\right) = \frac{1}{3}\cos\frac{3\pi}{2} = 0$, $f(x_{n-1}) = f\left(\frac{1}{2}\right) = \frac{1}{2}\cos\pi = -\frac{1}{2}$,

$f(x_n) = f(1) = \cos\frac{\pi}{2} = 0$

$$V_0^1(f)=\sup_{P\in P[0,1]}\left\{\begin{array}{l}\left|\frac{(-1)^n}{2n}-0\right|+\left|0-\frac{(-1)^n}{2n}\right|+\left|\frac{(-1)^{n+1}}{2n-2}-0\right|+\\ \ldots\ldots+\left|-\frac{1}{2}-0\right|+\left|0-\frac{1}{2}\right|\end{array}\right\}$$

$$=\sup_{P\in P[0,1]}\left\{\frac{1}{2n}+\frac{1}{2n}+\frac{1}{2n-2}+\frac{1}{2n-2}\ldots\ldots+\frac{1}{2}+\frac{1}{2}\right\}$$

$$=\sup_{P\in P[0,1]}\left\{\frac{1}{n}+\frac{1}{n-1}+\ldots\ldots+\frac{1}{2}+1\right\}$$

We know that $\sum\frac{1}{n}$ is divergent

$\therefore$ its partial sum $S_n=1+\frac{1}{2}+\ldots\ldots\ldots+\frac{1}{n}$ is not bounded above

$\therefore V_0^1(f)\to\infty$

$\therefore$ f is not function of bounded variation.

Example2.1.5. Give example to show that a continuous function may not be function of bounded variation.

Sol. Let $f(x)=\begin{cases} x\sin\left(\frac{\pi}{x}\right) & , \quad 0<x\le 1 \\ 0 & , \quad x=0\end{cases}$

First we prove f is continuous on [0.1]

In $0<x\le 1$, clearly f is continuous

At x = 0, f(0) = 0 , by definition for $|x-0|<\delta$

$$|f(x)-f(0)| = \left| x\sin\left(\frac{\pi}{x}\right) - 0 \right| = |x| \left| \sin\left(\frac{\pi}{x}\right) \right| = |x| < \delta = \varepsilon$$

$\therefore$ f is continuous on [0.1]

Next we prove f is not function of bounded variation.

Let a partition of [0, 1] such that $P = \left\{ 0, \frac{2}{2n+1}, \frac{2}{2n-1}, \frac{2}{2n-3}, \ldots.., \frac{2}{5}, \frac{2}{3}, 1 \right\}$

By definition $V_0^1(f)$ = $\sup\limits_{P \in P[0,1]} \sum\limits_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

$$= \sup_{P \in P[0,1]} \left\{ \begin{array}{l} |f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + |f(x_3) - f(x_2)| + \\ \ldots\ldots\ldots\ldots + |f(x_n) - f(x_{n-1})| \end{array} \right\}$$

$f(x_0) = f(0) = 0$, $f(x_1) = f\left(\frac{2}{2n+1}\right) = \frac{2}{2n+1}\sin\frac{(2n+1)\pi}{2} = \frac{(-1)^n 2}{2n+1}$,

$$f(x_2) = f\left(\frac{2}{2n-1}\right) = \frac{2}{2n-1}\sin\frac{(2n-1)\pi}{2} = \frac{2}{2n-1}\sin\left(n\pi - \frac{\pi}{2}\right) = \frac{2(-1)^{n+1}}{2n-1}$$

$$f(x_3) = f\left(\frac{2}{2n-3}\right) = \frac{2}{2n-3}\sin\frac{(2n-3)\pi}{2} = \frac{2(-1)^n}{2n-3}$$

$f(x_{n-2}) = f\left(\frac{2}{5}\right) = \frac{2}{5}\sin\frac{5\pi}{2} = \frac{2}{5}$, $f(x_{n-1}) = f\left(\frac{2}{3}\right) = \frac{2}{3}\sin\frac{3\pi}{2} = -\frac{2}{3}$,

$f(x_n) = f(1) = \sin\pi = 0$

$V_0^1(f)$

$$= \sup_{P \in P[0,1]} \left\{ \begin{array}{l} \left|\frac{(-1)^n 2}{2n+1} - 0\right| + \left|\frac{2(-1)^{n+1}}{2n-1} - \frac{(-1)^n 2}{2n+1}\right| + \left|\frac{(-1)^n 2}{2n-3} - \frac{2(-1)^{n+1}}{2n-1}\right| \\ \ldots\ldots + \left|-\frac{2}{3} - \frac{2}{5}\right| + \left|0 + \frac{2}{3}\right| \end{array} \right.$$

$$= \sup_{P \in P[0,1]} \left\{\frac{4}{2n+1} + \frac{4}{2n-1} + \ldots\ldots + \frac{4}{5} + \frac{4}{3}\right\}$$

$$= 4 \sup_{P \in P[0,1]} \left\{\frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2n+1}\right\}$$

We know that $\frac{1}{3} + \frac{1}{5} + \ldots\ldots$ is divergent

$\therefore$ its partial sum $s_n = \frac{1}{3} + \frac{1}{5} + \ldots\ldots + \frac{1}{2n-1}$ is not bounded above

$\therefore V_0^1(f) \to \infty$

$\therefore$ f is not function of bounded variation.

Note: It may also be seen that a function of bounded variation is not necessary continuous

The function $f(x) = [x]$,where $[x]$ denotes the greatest integer not greater than x, is a function of bounded variation on $[0,2]$ but is not continuous.

Theorem 2.1.3. Prove that a function of bounded variation is bounded.

Proof. Let $f : [a,b] \to R$ be a bounded variation.

$\therefore V_a^b(f) = \sup\limits_{P \in P[a,b]} \sum\limits_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$ = finite

$\therefore V_a^b(f) \le k$

T.P f(x) is bounded

$$|f(x)| = |f(x) - f(a) + f(a)| \le |f(x) - f(a)| + |f(a)| \le V_a^b(f) + |f(a)| \le k + |f(a)|$$

($\because V_a^b(f)$ is supremum and $|f(x) - f(a)|$ is one term of $V_a^b(f)$)

$\therefore$ f(x) is bounded on [a, b].

Note: Converse of above theorem is not true i.e. a bounded function may not be a function of bounded variation

Example 2.1.6. Let $f(x) = \begin{cases} x \sin\left(\dfrac{\pi}{x}\right), & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$

$|f(x)| = |x| \left| \sin \dfrac{\pi}{x} \right| \le 1$ i.e. f is bounded but f is not function of bounded variation.

Note. If f is not bounded then f is not function of bounded variation.

Example 2.1.7. Show $f(x) = \tan \dfrac{\pi x}{2}$, $x \in [-1,1]$ is not a function of a bounded variation.

Sol. $f(x) = \tan \dfrac{\pi x}{2}$

$\underset{x \to 1}{Lt} \tan \frac{\pi x}{2} = \infty \Rightarrow$ f is not bounded in $[-1,1]$

$\Rightarrow$ f is not a function of a bounded variation.

Example 2.1.8. Give an example of a function $f:[a,b] \to R$ which is bounded variation on every closed subinterval of (a, b) but it fails to be of bounded variation on [a, b].

Sol. Let $f:[0,1] \to R$ defined as $f(x) = \begin{cases} \frac{1}{x-1} & , \quad x \neq 1 \\ 0 & , \quad x = 1 \end{cases}$

This function is monotonically increasing on (0, 1) and therefore on every closed subinterval of (0, 1)

$\therefore$ it is of bounded variation on every closed subinterval of (0, 1)

However, because it has a vertical asymptote at x = 1, we can make the sum $\sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$ as large as we like by choosing partition points closed to 1.

Thus $V_0^1(f) = \infty$

Hence f is not bounded variation on [0, 1].

Example 2.1.9. Show that the function defined by

$f(x) = \begin{cases} \sqrt[3]{x} \sin\left(\frac{\pi}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$ is not of bounded variation on [0, 1]

Sol. Let $P = \left\{0, \frac{2}{2n+1}, \frac{2}{2n-1}, \frac{2}{2n-3}, \ldots\ldots, \frac{2}{5}, \frac{2}{3}, 1\right\}$ $f(x_0) = f(0) = 0$

$$f(x_1)=f\left(\frac{2}{2n+1}\right)=\left(\frac{2}{2n+1}\right)^{1/3}\sin(2n+1)\frac{\pi}{2}$$

$$=\left(\frac{2}{2n+1}\right)^{1/3}\cos n\pi=(-1)^n\left(\frac{2}{2n+1}\right)^{1/3}$$

$$f(x_2)=f\left(\frac{2}{2n-1}\right)=\left(\frac{2}{2n-1}\right)^{1/3}\sin(2n-1)\frac{\pi}{2}$$

$$=-\left(\frac{2}{2n-1}\right)^{1/3}\cos n\pi=(-1)^{n+1}\left(\frac{2}{2n-1}\right)^{1/3}$$

……………………………………………………………………………………

$$f(x_{n-2})=f\left(\frac{2}{5}\right)=\left(\frac{2}{5}\right)^{1/3}\sin\frac{5\pi}{2}=\left(\frac{2}{5}\right)^{1/3}\sin\left(2\pi+\frac{\pi}{2}\right)=\left(\frac{2}{5}\right)^{1/3}$$

$$f(x_{n-1})=f\left(\frac{2}{3}\right)=\left(\frac{2}{3}\right)^{1/3}\sin\frac{3\pi}{2}=-\left(\frac{2}{3}\right)^{1/3},$$

$$f(x_n)=f(1)=(1)^{1/3}\sin\pi=0$$

$$V_0^1(f)=\sup_P\sum_{i=1}^{n}\left|f(x_i)-f(x_{i-1})\right|$$

$$= 2\times 2^{1/3}\left\{\frac{1}{3^{1/3}}+\frac{1}{5^{1/3}}+\dots\dots+\frac{1}{(2n+1)^{1/3}}\right\}= \text{not finite}$$

$$\because\ a_n=\frac{1}{(2n+1)^{1/3}},\ b_n=\frac{1}{n^{1/3}}\Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^{1/3}}{(2n+1)^{1/3}}=\frac{1}{2}$$

But by p- test b_n is divergent

$\therefore\ \sum a_n$ is divergent.

Example 2.1.10. Show that the function defined as $f(x)=\begin{cases}1 & , \quad x\in Q' \\ 0 & , \quad x\in Q\end{cases}$ is not bounded variation on any interval.

Sol. Let [a, b] be any arbitrary closed interval in R. Let $P=\{a=x_0, x_1, x_2, \ldots\ldots, x_{2n+1}, x_{2n+2}=b\}$ be any partition of [a, b] . As we know that between two real numbers there is rational number and irrational number. Take x_1 to be irrational number between a and b. Take x_2 be an rational number between x_1 and b. continue like this taking x_{2n+1} to be an irrational number between x_{2n} and b and finally $x_{2n+2}=b$.Then

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|=|f(x_1)-f(x_0)|+|f(x_2)-f(x_1)|$$

$$+|f(x_3)-f(x_2)|+\ldots\ldots+|f(x_{2n+1})-f(x_{2n})|+|f(x_{2n+2})-f(x_{2n+1})|$$

$$\geq|f(x_2)-f(x_1)|+|f(x_3)-f(x_2)|+\ldots\ldots+|f(x_{2n+1})-f(x_{2n})| = 2\text{ n}$$

Thus $V_a^b(f)\to\infty$

$\therefore$ f is not bounded variation.

Example 2.1.11. Show that $f(x)=\sin x$ in $\left[0,\frac{\pi}{2}\right]$, Is a function of bounded variation ? If so the total variation?

Sol. As sin x is monotonically increasing in $\left[0,\frac{\pi}{2}\right]$ $\Rightarrow$ f(x) is of bounded variation as in $\left[0,\frac{\pi}{2}\right]$

Then $V\left(f,0,\frac{\pi}{2}\right)=\sin\frac{\pi}{2}-\sin 0=1$

Example 2.1.12. Show that $f(x)=3x^4-56x^3+336x^2-768x+1100$ on [0, 10] is of bounded variation. Also find its total variation.

Sol. $f(x)=3x^4-56x^3+336x^2-768x+1100$

Then $f'(x)=12x^3-168x^2+672x-768$

$$|f'(x)|\le 12|x|^3+168|x|^2+672|x|+768$$

$$\Rightarrow |f'(x)|\le 12|x|^3+168|x|^2+672|x|+768\le 12000+16800+6720+768=36288$$

$\therefore$ f(x) is bounded variation on [0, 10]'

Total variation = $V(f,0,10)\le 36288(10-0)=362880$

Exercise 2.1

1.Show that $f(x)=\cos x$ is bounded variation over a $\left[0,\dfrac{\pi}{2}\right]$

2. If $f(x)=\begin{cases} x^p\sin\dfrac{1}{x}, & x\ne 0 \\ 0, & x=0\end{cases}$, $p\ge 2$ show that f is of bounded variation on $[-1,1]$

3. if f is constant on [a, b], then prove that f is of bounded variation on [a, b]

4. Show that $f(x)=x^2+x+1$ on [- 1, 1] is a function of a bounded variation.

5. Show $f(x)=\sqrt{1-x^2}$,on [-1, 1] is a function of a bounded variation.

6. Show that a polynomial function f is of bounded variation on every closed and bounded interval [a, b].

7.Prove that $f(x)=\begin{cases} \sqrt[2]{x}\sin\left(\frac{1}{x}\right) & , \quad x\neq 0 \\ 0 \quad , & x=0 \end{cases}$ is not of bounded variation on [0, 1].

2.2. Some Properties of Functions of Bounded Variation.

Theorem 2.2.1. The sum (difference) of two functions of bounded variation is also bounded variation. Also prove that

$$V_a^b(f+g)\leq V_a^b(f)+V_a^b(g)$$

Proof.

$$V_a^b(f+g) = \sup_{P\in P[a,b]} \sum_{i=1}^{n} \left|(f+g)(x_i) - (f+g)(x_{i-1})\right|$$

$$= \sup_{P\in P[a,b]} \sum_{i=1}^{n} \left|[f(x_i)+g(x_i)] - [f(x_{i-1})+g(x_{i-1})]\right|$$

$$= \sup_{P\in P[a,b]} \sum_{i=1}^{n} \left|[f(x_i)-f(x_{i-1})] + [g(x_i)-g(x_{i-1})]\right|$$

$$\leq \sup_{P\in P[a,b]} \sum_{i=1}^{n} \left|f(x_i)-f(x_{i-1})\right| + \sup_{P\in P[a,b]} \sum_{i=1}^{n} \left|g(x_i)-g(x_{i-1})\right| =$$

$$V_a^b(f) + V_a^b(g) < \infty$$

$\therefore$ f + g function of bounded variation.

Similarly it may be shown that f – g is of bounded variation over [a, b] and

$$V_a^b(f-g) \leq V_a^b(f) + V_a^b(g)$$

Theorem2.2.2. Let f be function of bounded variation defined on [a, b] and c be a constant then $V_a^b(cf) = |c| V_a^b(f)$

Proof.

$$V_a^b(cf) = \sup_{P\in P[a,b]} \sum_{i=1}^{n} \left|(cf)(x_i) - (cf)(x_{i-1})\right| = \sup_{P\in P[a,b]} \sum_{i=1}^{n} \left|cf(x_i) - cf(x_{i-1})\right|$$

$$= |c| \sup_{P\in P[a,b]} \sum_{i=1}^{n} \left|f(x_i) - f(x_{i-1})\right| = |c| V_a^b(f)$$

Theorem2.2.3. The product of two functions of bounded variation is also bounded variation.

Proof. Let f and g being of bounded variation over [a,b], are both bounded and accordingly there exists a number k such that $\forall\ x \in [a,b]$

$|f(x)| \le k$, $|g(x)| \le k$

$$V_a^b(fg) = \sup_{P \in P[a,b]} \sum_{i=1}^{n} |(fg)(x_i) - (fg)(x_{i-1})|$$

$$= \sup_{P \in P[a,b]} \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})|$$

$$= \sup_{P \in P[a,b]} \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_i) + f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1})|$$

$$= \sup_{P \in P[a,b]} \sum_{i=1}^{n} |[f(x_i) - f(x_{i-1})]g(x_i) + [g(x_i) - g(x_{i-1})]f(x_{i-1})|$$

$$\le \sup_{P \in P[a,b]} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})||g(x_i)| + |g(x_i) - g(x_{i-1})||f(x_{i-1})|$$

$$\le k \sup_{P \in P[a,b]} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + k \sup_{P \in P[a,b]} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

$$= kV_a^b(f) + kV_a^b(g) < \infty$$

$\therefore$ f g is function of bounded variation

$\therefore V_a^b(fg) \le kV_a^b(f) + kV_a^b(g)$

Theorem2.2.4 If f is a function of bounded variation on [a, b] and if there exists a positive number k such that $|f(x)| \ge k$,

$\forall x \in [a,b]$, then 1/f is also of bounded variation on [a, b].

Proof.

$$V_a^b\left(\frac{1}{f}\right) = \sup_{P \in P[a,b]} \sum_{i=1}^{n} \left|\left(\frac{1}{f}\right)(x_i) - \left(\frac{1}{f}\right)(x_{i-1})\right| = \sup_{P \in P[a,b]} \sum_{i=1}^{n} \left|\frac{1}{f(x_i)} - \frac{1}{f(x_{i-1})}\right|$$

$$= \sup_{P \in P[a,b]} \sum_{i=1}^{n} \left|\frac{f(x_{i-1}) - f(x_i)}{f(x_i)f(x_{i-1})}\right| = \sup_{P \in P[a,b]} \sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|}{|f(x_i)||f(x_{i-1})|}$$

$\because |f(x)| \geq k,\ \forall x \in [a,b]$

$\therefore \dfrac{1}{|f(x_i)|} \leq \dfrac{1}{k}, \dfrac{1}{|f(x_{i-1})|} \leq \dfrac{1}{k}$

$$V_a^b\left(\frac{1}{f}\right) \leq \frac{1}{k^2} \sup_{P \in P[a,b]} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \frac{1}{k^2} V_a^b(f) < \infty$$

$$\therefore V_a^b\left(\frac{1}{f}\right) \leq \frac{1}{k^2} V_a^b(f)$$

Note: If f and g are functions of bounded variation on [a, b] and $|g(x)| \geq k$ $\forall x \in [a,b]$ for some positive real number k, then f/g is also bounded variation on [a, b].

Since g is bounded variation on [a, b] , therefore 1/g is a bounded variation on [a, b].

Also f is a function of bounded variation on [a, b]

$\therefore$ f/g is a function of bounded variation on [a, b]

Theorem2.2.5. If f is a function of bounded variation on [a, b] then it is also of bounded variation on [a, c] and [c, b] where c is a point of [a, b] and conversely. Also $V_a^b(f) = V_a^c(f) + V_c^b(f)$

Proof. Let first f be of bounded variation on [a, b].

Let $P_1 = \{a = x_0, x_1, x_2, ..., x_m = c\}$ and $P_2 = \{c = x_m, x_{m+1}, x_{m+2}, ..., x_{m+n} = b\}$ be any two partition of [a, c] and [c, b] respectively.

They give rise to a partition $P = P_1 \cup P_2 = \{a = x_0, x_1, x_2, ..., x_m, x_{m+1}, x_{m+2}, ..., x_{m+n} = b\}$ of[a, b].

$$\sum_{i=1}^{m+n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| + \sum_{i=m+1}^{m+n} |f(x_i) - f(x_{i-1})|$$

Since each term is no – negative so

$$\sum_{i=1}^{m}\left|f(x_i)-f(x_{i-1})\right| \leq \sum_{i=1}^{m+n}\left|f(x_i)-f(x_{i-1})\right|$$

$$\sup_{P\in P[a,c]}\sum_{i=1}^{m}\left|f(x_i)-f(x_{i-1})\right| \leq \sup_{P\in P[a,b]}\sum_{i=1}^{m+n}\left|f(x_i)-f(x_{i-1})\right|$$

$$\therefore V_a^c(f) \leq V_a^b(f) < \infty$$

Also $\sup_{P\in P[c,b]}\sum_{i=m}^{m+n}\left|f(x_i)-f(x_{i-1})\right| \leq \sup_{P\in P[a,b]}\sum_{i=1}^{m+n}\left|f(x_i)-f(x_{i-1})\right|$

$$\therefore V_c^b(f) \leq V_a^b(f) < \infty$$

$\therefore$ f is of bounded variation on [a, c] and [c, b] both.

Let, now, f is of bounded variation on [a, c] and [c, b] both.

Let $P = \{a = z_0, z_1, z_2, ..., z_{r-1}, z_r ..., z_n = b\}$ be any partition of [a, b].

Let us consider the partition $P^* = P \cup \{c\}$, let $z_{r-1} \leq c \leq z_r$

We have $V_a^b(f) = \sup_{P\in P[a,b]}\sum_{i=1}^{n}\left|f(z_i)-f(z_{i-1})\right|$

$$= \sup_{P\in P[a,b]}\left\{\sum_{i=1}^{r-1}\left\{\left|f(z_i)-f(z_{i-1})\right|\right\}+\left|f(z_r)-f(z_{r-1})\right|+\sum_{i=r+1}^{n}\left|f(z_i)-f(z_{i-1})\right|\right\}$$

$$= \sup_{P\in P[a,b]}\left\{\sum_{i=1}^{r-1}\left\{\left|f(z_i)-f(z_{i-1})\right|\right\}+\left|f(z_r)-f(c)+f(c)-f(z_{r-1})\right|+\sum_{i=r+1}^{n}\left|f(z_i)-f\right.\right.$$

$$\leq \sup_{P\in P[a,b]}\left\{\sum_{i=1}^{r-1}\left\{\left|f(z_i)-f(z_{i-1})\right|+\left|f(c)-f(z_{r-1})\right|\right\}+\left\{\left|f(z_r)-f(c)\right|+\sum_{i=r+1}^{n}\left|f(z_i)-f\right.\right.\right.$$

$= V_a^c(f) + V_c^b(f) < \infty$ {$\because$ f is of bounded variation on [a, c] and [c, b] both }

$\Rightarrow$ f is of bounded variation on [a, b] and also

$V_a^b(f) \leq V_a^c(f) + V_c^b(f)$..............................(i)

Let, now, f is of bounded variation on [a, c] and [c, b] both.

Let $P_1 = \{a = x_0, x_1, x_2, \ldots\ldots, x_m = c\}$ and $P_2 = \{c = y_0, y_1, y_2, \ldots\ldots, y_n = b\}$ be any two partition
of [a, c] and [c, b] respectively.
By definition of l.u.b.

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| > V_a^c(f) - \frac{\varepsilon}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

and $\displaystyle\sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| > V_c^b(f) - \frac{\varepsilon}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$

Adding (ii) and (iii) we get

$$\sum_{i=1}^{m} \left| f(x_i) - f(x_{i-1}) \right| + \sum_{i=1}^{n} \left| f(y_i) - f(y_{i-1}) \right| > V_a^c(f) + V_c^b(f) - \varepsilon$$

$$\therefore V_a^b(f) > V_a^c(f) + V_c^b(f) - \varepsilon$$

But since ε is an arbitrary positive number, we get

$$V_a^b(f) \geq V_a^c(f) + V_c^b(f) \ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

From (i) and (iv) we get $V_a^b(f) = V_a^c(f) + V_c^b(f)$

2. 3. Definition. Variation Function or total variation function of [a, x] as a function of x

Let $f:[a,b]\to R$ be a function of bounded variation. Then the function $V:[a,b]\to R$ defined by

$V(x)=\begin{cases} V_a^x(f) & , \quad a<x\le b \\ 0 & , \quad x=a \end{cases}$ is called the variation function.

Theorem 2.3.1. Let $f:[a,b]\to R$ be a function of bounded variation. Then the function $V:[a,b]\to R$ defined by

$$V(x)=\begin{cases} V_a^x(f) & , \quad a<x\le b \\ 0 & , \quad x=a \end{cases}$$

Then (i) V is monotonically increasing on [a, b]

(ii) V – f is monotonically increasing on [a, b]

Proof. (i) Let $a<x<y\le b$

When $a<x\le b$ then $V(x)=V_a^x(f)$ and when $a<y\le b$ then $V(y)=$

$V_a^y(f)$

$V_a^y(f)=V_a^x(f)+V_x^y(f)$

$\Rightarrow$ V(y) - V (x) = $V_x^y(f)\ge 0$ …………(i)

[$\because V_x^y(f)=\sup\limits_{P\in P[x,y]}\sum\limits_{i=1}^{n}\left|f(t_i)-f(t_{i-1})\right|\ge 0$]

$\Rightarrow V(y)\ge V(x)$ Whenever y > x

$\therefore$ V is monotonically increasing on [a, b]

(ii) Let D(x) = V(x) – f(x)

$D(y)-D(x)=[V(y)-f(y)]-[V(x)-f(x)]$

$=[V(y)-V(x)]-[f(y)-f(x)]$

$=V_x^y(f)-[f(y)-f(x)]$ From (i)

Also from definition of $V_x^y(f)$

$f(y)-f(x)\le V_x^y(f)$

$\therefore\ D(y)-D(x)\ge 0\Rightarrow D(y)\ge D(x)$

Hence D = V – f is monotonically increasing on [a, b]

2.4. Functions of bounded variation expressed as the difference of increasing functions.

Theorem 2.4.1. (Jordan Decomposition Theorem) A function f is of bounded variation on [a, b] if and only if f is difference of two monotonically increasing real valued functions defined on [a, b].

Proof. Let f be a function of bounded variation on [a ,b]

$f=\frac{1}{2}(2f)=\frac{1}{2}\{(V+f)-(V-f)\}$ where V is **Variation Function.**

$=\frac{1}{2}(V+f)-\frac{1}{2}(V-f)=g-h$ where $g=\frac{1}{2}(V+f), h=\frac{1}{2}(V-f)$

$\therefore$ f = g – h

T.P. g and h are monotonically increasing

For this let $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$

T.P. $g(x_1) \le g(x_2), h(x_1) \le h(x_2)$

$$g(x_2)-g(x_1)=\frac{1}{2}\{V(x_2)+f(x_2)\}-\frac{1}{2}\{V(x_1)+f(x_1)\}$$

$$=\frac{1}{2}\{V(x_2)-V(x_1)\}+\frac{1}{2}\{f(x_2)-f(x_1)\} \quad \text{..................(i)}$$

$$h(x_2)-h(x_1)=\frac{1}{2}\{V(x_2)-f(x_2)\}-\frac{1}{2}\{V(x_1)-f(x_1)\}$$

$$=\frac{1}{2}\{V(x_2)-V(x_1)\}-\frac{1}{2}\{f(x_2)-f(x_1)\} \quad \text{.....................(ii)}$$

Now f is a function of bounded variation on [a. b]

$\therefore$ f is a function of bounded variation on $[x_1, x_2]$

$\therefore |f(x_2)-f(x_1)| \le V_{x_1}^{x_2}(f)$

$= V_a^{x_1}(f) + V_{x_1}^{x_2}(f) - V_a^{x_1}(f)$

$= V_a^{x_2}(f) - V_a^{x_1}(f) = V(x_2) - V(x_1)$ {by definition of variation function}

$\therefore f(x_2)-f(x_1) \le V(x_2)-V(x_1)$ and $-\{f(x_2)-f(x_1)\} \le V(x_2)-V(x_1)$

$\therefore \frac{1}{2}\{V(x_2)-V(x_1)\} - \frac{1}{2}\{f(x_2)-f(x_1)\} \ge 0$ and

$\frac{1}{2}\{V(x_2)-V(x_1)\} + \frac{1}{2}\{f(x_2)-f(x_1)\} \ge 0$

From (i) and (ii)

$h(x_2)-h(x_1) \ge 0$ and $g(x_2)-g(x_1) \ge 0$

$\Rightarrow h(x_1) \le h(x_2),\ g(x_1) \le g(x_2)$

Hence g and h are monotonically increasing.

Conversely, let f = g – h where g and h are monotonically increasing functions defined on [a, b]

T.P. f is function of bounded variation on [a, b].

g and h are functions of bounded variation on [a, b]

By property of bounded variation

$\Rightarrow$ g – h is a function of bounded variation on [a, b]

$\Rightarrow$ f is a function of bounded variation on [a, b].

Example 2.4.1. Represent $f(x) = \cos^2 x,\ 0 \le x \le 2\pi$ as a difference of two increasing functions.

Sol. By Jordan decomposition theorem, we have f = g – h where

$g = \frac{1}{2}(V + f), h = \frac{1}{2}(V - f)$

For this we divide $[0,2\pi]$ into four sub- intervals $\left[0,\frac{\pi}{2}\right],\left[\frac{\pi}{2},\pi\right],\left[\pi,\frac{3\pi}{2}\right],$ $\left[\frac{3\pi}{2},2\pi\right]$

In $\left[0,\frac{\pi}{2}\right]$, f(x) decrease

$$V(x)=V(f,0,x)=f(0)-f(x)=1-\cos^2 x \Rightarrow V\left(f,0,\frac{\pi}{2}\right)=1$$

In $\left[\frac{\pi}{2},\pi\right]$, f(x) increase

$$V\left(f,\frac{\pi}{2},x\right)=f(x)-f\left(\frac{\pi}{2}\right)=\cos^2 x$$

$$\Rightarrow V(x)=V(f,0,x)=V\left(f,0,\frac{\pi}{2}\right)+V\left(f,\frac{\pi}{2},x\right)=1+\cos^2 x \therefore V(f,0,\pi)=1+1$$

In $\left[\pi,\frac{3\pi}{2}\right]$, f(x) decrease

$$V(f,\pi,x)=f(\pi)-f(x)=1-\cos^2 x$$

$$\Rightarrow V(x)=V(f,0,x)=V(f,0,\pi)+V(f,\pi,x)=2+1-\cos^2 x=3-\cos^2 x$$

In $\left[\frac{3\pi}{2},2\pi\right]$, f(x) increase

$$V\left(f,\frac{3\pi}{2},x\right)=f(x)-f\left(\frac{3\pi}{2}\right)=\cos^2 x$$

$$\Rightarrow V(x)=V(f,0,x)=V\left(f,0,\frac{3\pi}{2}\right)+V\left(f,\frac{3\pi}{2},x\right)=3+\cos^2 x$$

$$\therefore V(x)=\begin{cases} 1-\cos^2 x \quad , \quad 0\le x\le \dfrac{\pi}{2} \\ 1+\cos^2 x \quad , \quad \dfrac{\pi}{2}\le x\le \pi \\ 3-\cos^2 x, \quad , \quad \pi\le x\le \dfrac{3\pi}{2} \\ 3+\cos^2 x \quad , \quad \dfrac{3\pi}{2}\le x\le 2\pi \end{cases}$$

$$\therefore\ g=\begin{cases} 1/2 \quad , \quad 0\le x\le \dfrac{\pi}{2} \\ 1/2+\cos^2 x \quad , \quad \dfrac{\pi}{2}\le x\le \pi \\ 3/2, \quad , \quad \pi\le x\le \dfrac{3\pi}{2} \\ 3/2+\cos^2 x \quad , \quad \dfrac{3\pi}{2}\le x\le 2\pi \end{cases}$$ and

$$h=\begin{cases} 1/2-\cos^2 x \quad , \quad 0\le x\le \dfrac{\pi}{2} \\ 1/2 \quad , \quad \dfrac{\pi}{2}\le x\le \pi \\ 3/2-\cos^2 x, \quad , \quad \pi\le x\le \dfrac{3\pi}{2} \\ 3/2 \quad , \quad \dfrac{3\pi}{2}\le x\le 2\pi \end{cases}$$

Example 2.4.2 Represent $f(x)=[x]-x, 0\le x\le 2$ as a difference of two increasing functions.

Sol. $f(x)=\begin{cases} -x, & 0\le x<1 \\ 1-x, & 1\le x<2 \\ 0, & x=1,2 \end{cases}$

In $[0,1)$ f(x) decrease

V(x) = V(f, 0, x) = f(0) - f(x) = x = [x] + x

$\therefore V(f,0,1) = \sup\limits_{0<x<1} \{V(f,0,x)+V(f,x,1)\}$

$= \sup\limits_{0<x<1} \{x+|f(1)-f(x)|\} = \sup\limits_{0<x<1} \{x+|0+x|\} = 2$

$\therefore [v(x)]_{x=1} = 2$ =[x] + x

In $[1,2)$ f(x) decrease

$V(f,1,x) = f(1) - f(x) = 0-1+x = -1+x$

$\therefore$ V(x) = $V(f,0,x) = V(f,0,1)+V(f,1,x) = 2-1+x = 1+x$ = [x] + x

$\therefore V(f,1,2) = \sup\limits_{1<x<2} \{V(f,1,x)+V(f,x,2)\}$

$= \sup\limits_{1<x<2} \{-1+x+|f(2)-f(x)|\} = \sup\limits_{1<x<2} \{-1+x+|0-1+x|\} = 2 \therefore$

$[v(x)]_{x=2} = V(f,0,2) = V(f,0,1)+V(f,1,2) = 2+2 = 4$ = [x] + x

Thus V(x) = [x] + x for $0 \le x \le 2$

$\therefore g = \frac{1}{2}(v+f) = \frac{1}{2}([x]+x+[x]-x) = [x]$ and

$h = \frac{1}{2}(v-f) = \frac{1}{2}([x]+x-[x]+x) = x$

Exercise 2. 4.

1.Represent $f(x) = \begin{cases} -x^2, & 0 \le x < 1 \\ 0, & x = 1 \\ 1, & 1 < x \le 2 \end{cases}$ as a difference of two increasing functions.

2.5. Continuous functions of bounded variation.

Theorem 2.5.1. A function f of bounded variation on [a, b] is continuous if and only if V is continuous.

Proof. Firstly suppose that V is continuous on [a, b]. Let $c \in [a,b]$

There exist $\delta > 0$ such that $|V(x) - V(c)| < \in$ for $|x - c| < \delta$

T.P. f is continuous at x = c.

$$|f(x) - f(c)| \le \left|V_c^x(f)\right| = \left|V_a^c(f) + V_c^x(f) - V_a^c(f)\right| = \left|V_a^x(f) - V_a^c(f)\right|$$

$$= |V(x) - V(c)| < \in$$

$\therefore$ f is continuous at x = c.

Conversely, suppose that f is continuous on [a, b].

Let $c \in [a,b]$

There exist $\delta > 0$ such that $|f(x) - f(c)| < \frac{\in}{2}$ for $|x - c| < \delta$

Now f is a function of bounded variation on [a, b]

$\Rightarrow$ f is function of bounded variation on [c, b]

Let $P = \{c = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ partition of [c, b] such that

$$\sum_{i=1}^{n} \left|f(x_i) - f(x_{i-1})\right| > V_c^b(f) - \frac{\in}{2} \qquad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Assume that the first subinterval $[x_0, x_1]$ in P of length less than δ.

From (i)

$$V_c^b(f) - \frac{\in}{2} < \sum_{i=1}^{n} \left|f(x_i) - f(x_{i-1})\right|$$

$$= \left|f(x_1) - f(x_0)\right| + \sum_{i=2}^{n} \left|f(x_i) - f(x_{i-1})\right|$$

$$< \frac{\in}{2} + \sum_{i=2}^{n} \left|f(x_i) - f(x_{i-1})\right| < \frac{\in}{2} + V_{x_1}^b(f)$$

$V_c^b(f) - V_{x_1}^b(f) < \varepsilon$

$\Rightarrow V(x) - V(c) < \varepsilon$ for $|x-c| < \delta$

$\Rightarrow \lim\limits_{x \to c^+} V(x) = V(c)$

Similarly, it can be shown that $\lim\limits_{x \to c^-} V(x) = V(c)$

$\therefore$ V is continuous at x = c.

Hence V is continuous on [a, b].

Cor. A continuous function is of bounded variation if and only ifit can express as a difference of two continuous monotonically increasing functions.

Proof. Let f be a continuous function of bounded variation.

Also, V is continuous.

$\therefore\ g = \frac{1}{2}(V + f)$ and $h = \frac{1}{2}(V - f)$ are continuous and f = g – h

Also g and h are monotonically increasing functions.

Hence f is difference of two continuous monotonically increasing functions.

Conversely, let f = g – h , where g and h are continuous monotonically increasing functions.

Since g and h are continuous functions, therefore f = g – h is also continuous.

Also by Jordan decomposition theorem, f is of bounded variation.

Hence f is a continuous function of bounded variation.

2.6.Positive and Negative Variations

For any $a \in R$ set $a^{+} = \max\{a,0\}$ and $a^{-} = \max\{-a,0\}$

We begin by noticing the following equalities:

$a^{+} + a^{-} = |a|$ and $a^{+} - a^{-} = a$ ....(i)

If $a > 0$ then $a^{+} = a = |a|$ and $a^{-} = 0$

$\Rightarrow a^{+} + a^{-} = |a|$ and $a^{+} - a^{-} = a$

If $a < 0$ then $a^{+} = 0$ and $a^{-} = -a \Rightarrow a^{+} + a^{-} = -a = |a|$ and $a^{+} - a^{-} = a$

from (i) we have $a^{+} = \dfrac{a+|a|}{2}$ Take $a = b + c$

$$(b+c)^{+} = \frac{b+c+|b+c|}{2} \le \frac{b+c+|b|+|c|}{2} = \frac{b+|b|}{2} + \frac{c+|c|}{2} = b^{+} + c^{+} \therefore$$

$$(b+c)^{+} \le b^{+} + c^{+}$$

Definition 2.6.1. Let f be a real valued function defined on [a, b] and
Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ partition of [a, b].
Let $A(P) = \{i : f(x_i) - f(x_{i-1}) > 0\}$ and $B(P) = \{i : f(x_i) - f(x_{i-1}) < 0\}$
Then

$$p_f(a,b) = \sup_P \left\{ \sum_{i \in A(P)} [f(x_i) - f(x_{i-1})] \right\}$$ and

$$n_f(a,b) = \sup_P \left\{ \sum_{i \in B(P)} |f(x_i) - f(x_{i-1})| \right\}$$ are called positive and negative

variations of f on [a, b]

Note. $p_f(a,b) \ge 0$ and $n_f(a,b) \ge 0$

Theorem 2.6.1. If f is a function of bounded variation on [a, b], then

$p_f(a,b) - n_f(a,b) = f(b) - f(a)$ and

$V_f(a,b) = p_f(a,b) + n_f(a,b)$

Proof. Let $P = \{c = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ partition of [a, b].

Then $\sum_{i \in A(P)} [f(x_i) - f(x_{i-1})] - \sum_{i \in B(P)} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$

$= f(b) - f(a)$

$$\Rightarrow \sum_{i \in A(P)} [f(x_i) - f(x_{i-1})] = \sum_{i \in B(P)} |f(x_i) - f(x_{i-1})| + f(b) - f(a) \quad \ldots\ldots\ldots \text{(i)}$$

$$\sup_P \sum_{i \in A(P)} [f(x_i) - f(x_{i-1})] = \sup_P \left\{ \sum_{i \in B(P)} |f(x_i) - f(x_{i-1})| \right\} + f(b) - f(a)$$

$\Rightarrow p_f(a,b) = n_f(a,b) + f(b) - f(a)$

We get $p_f(a,b) - n_f(a,b) = f(b) - f(a)$ ………………… (ii)

Next we prove $V_f(a,b) = p_f(a,b) + n_f(a,b)$

Again $\sum_{i \in A(P)} [f(x_i) - f(x_{i-1})] + \sum_{i \in B(P)} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

$$\leq \sup_{P \in P[a,b]} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$$

$$\Rightarrow \sup_{P \in P[a,b]} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right| \geq \sum_{i \in A(P)} \left[f(x_i) - f(x_{i-1}) \right] + \sum_{i \in B(P)} \left| f(x_i) - f(x_{i-1}) \right|$$

$$\Rightarrow V_f(a,b) \geq \sum_{i \in A(P)} \left[f(x_i) - f(x_{i-1}) \right] + \sum_{i \in B(P)} \left| f(x_i) - f(x_{i-1}) \right|$$

From (i)

$$\Rightarrow V_f(a,b) \geq \sum_{i \in A(P)} \left[f(x_i) - f(x_{i-1}) \right] - \left[f(b) - f(a) - \sum_{i \in A(P)} \left[f(x_i) - f(x_{i-1}) \right] \right]$$

$$\Rightarrow V_f(a,b) \geq 2 \sum_{i \in A(P)} \left[f(x_i) - f(x_{i-1}) \right] - p_f(a,b) + n_f(a,b)$$ From (ii)

$$\Rightarrow V_f(a,b) \geq 2 \sup_{P} \sum_{i \in A(P)} \left[f(x_i) - f(x_{i-1}) \right] - p_f(a,b) + n_f(a,b)$$

$$\Rightarrow V_f(a,b) \geq 2 p_f(a,b) - p_f(a,b) + n_f(a,b)$$

$$\Rightarrow V_f(a,b) \geq p_f(a,b) + n_f(a,b) \text{ .... (iii)}$$

Also, $V_f(a,b) = \sup_{P \in P[a,b]} \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|$

$$= \sup_{P \in P[a,b]} \left\{ \sum_{i \in A(P)} \left[f(x_i) - f(x_{i-1}) \right] + \left\{ \sum_{i \in B(P)} \left| f(x_i) - f(x_{i-1}) \right| \right\} \right\}$$

$$\leq \sup_{P \in P[a,b]} \left\{ \sum_{i \in A(P)} \left[f(x_i) - f(x_{i-1}) \right] \right\} + \sup_{P \in P[a,b]} \left\{ \sum_{i \in B(P)} \left| f(x_i) - f(x_{i-1}) \right| \right\}$$

$$\Rightarrow V_f(a,b) \leq p_f(a,b) + n_f(a,b) \text{ .... (iv)}$$

From (iii) and (iv) we get $V_f(a,b) = p_f(a,b) + n_f(a,b)$

Note. $p_f(a,b) = \frac{1}{2}\left[V_f(a,b) + f(b) - f(a)\right]$ and

$n_f(a,b) = \frac{1}{2}\left[V_f(a,b) - f(b) + f(a)\right]$

Example 2.6.1 Find $V_f(-2,x)$, $p_f(-2,x)$ and $n_f(-2,x)$

If $f(x) = 4x^3 - 3x^4$, $x \in [-2,2]$

Sol. $f'(x) = 12x^2 - 12x^3 = 12x^2(1-x) = -12x^2(x-1)$

$\therefore$ f(x) increasing on $[-2,1]$ and decreasing on $[1,2]$

For $x \in [-2,1]$ then $V_f(-2,x) = f(x) - f(-2) = 4x^3 - 3x^4 + 80$

$\therefore V_f(-2,1) = 81$

For $x \in [1,2]$ then $V_f(1,x) = f(1) - f(x) = 1 - 4x^3 + 3x^4$

$\therefore V_f(-2,x) = V_f(-2,1) + V_f(1,x) = 81 + 1 - 4x^3 + 3x^4 = 82 - 4x^3 + 3x^4$

$$\therefore V_f(-2,x) = \begin{cases} 4x^3 - 3x^4 + 80, & x \in [-2,1] \\ 82 - 4x^3 + 3x^4, & x \in [1,2] \end{cases}$$

For $x \in [-2,1]$, $p_f(-2,x) = \frac{1}{2}\left[V_f(-2,x) + f(x) - f(-2)\right]$

$= \frac{1}{2}\left[4x^3 - 3x^4 + 80 + 4x^3 - 3x^4 + 80\right] = 4x^3 - 3x^4 + 80$

$n_f(-2,x) = \frac{1}{2}\left[V_f(-2,x) - f(x) + f(-2)\right] = 0$

For $x \in [1,2]$, $p_f(1,x) = \frac{1}{2}\left[V_f(1,x) + f(x) - f(1)\right] = 0$ and

$n_f(1,x) = \frac{1}{2}\left[V_f(1,x) - f(x) + f(1)\right] = 1 - 4x^3 + 3x^4$

$\therefore\ p_f(-2,x) = p_f(-2,1) + p_f(1,x) = 81,$

$n_f(-2,x) = n_f(-2,1) + n_f(1,x) = 1 - 4x^3 + 3x^4$

Thus we have $p_f(-2,x) = \begin{cases} 4x^3 - 3x^4 + 80, & x \in [-2,1] \\ 81, & x \in (1,2] \end{cases}$

And $n_f(-2,x) = \begin{cases} 0, & x \in [-2,1] \\ 1 - 4x^3 + 3x^4, & x \in (1,2] \end{cases}$

Example 2.6.2 Find $V_f(-2,x)$, $p_f(-2,x)$ and $n_f(-2,x)$

If $f(x) = 3x^2 - 2x^3$, $x \in [-2,2]$

Sol. $f(x) = 3x^2 - 2x^3$

Then $f'(x) = 6x - 6x^2 = 6x(1-x) = -6x(x-1)$

For $f'(x) \ge 0 \Rightarrow -6x(x-1) \ge 0 \Rightarrow x(x-1) \le 0 \Rightarrow x \in [0,1]$

For $f'(x) \le 0 \Rightarrow -6x(x-1) \le 0 \Rightarrow x(x-1) \ge 0 \Rightarrow x \in [-2,0] \cup [1,2]$

For $x \in [-2,0]$

$V_f(-2,x) = f(-2) - f(x) = 28 - 3x^2 + 2x^3 = 28 - 3x^2 + 2x^3$

$\therefore V_f(-2,0) = 28$

For $x \in [0,1]$

$V_f(0,x) = f(x) - f(0) = 3x^2 - 2x^3$

$V_f(-2,x) = V_f(-2,0) + V_f(0,x) = 28 + 3x^2 - 2x^3$

$\therefore V_f(-2,1) = 29$

For $x \in [1,2]$

$V_f(1,x) = f(1) - f(x) = 1 - 3x^2 + 2x^3$

$V_f(-2,x) = V_f(-2,1) + V_f(1,x) = 30 - 3x^2 + 2x^3$

Thus the total variation function on $-2 \le x \le 2$ is defined as

$$V_f(-2,x) = \begin{cases} 28 - 3x^2 + 2x^3, & x \in [-2,0] \\ 28 + 3x^2 - 2x^3, & x \in (0,1] \\ 30 - 3x^2 + 2x^3, & x \in (1,2] \end{cases}$$

For $x \in [-2,0]$ $p_f(-2,x) = \frac{1}{2}\left[V_f(-2,x) + f(x) - f(-2)\right] = 0$ and

$n_f(-2,x) = \frac{1}{2}\left[V_f(-2,x) - f(x) + f(-2)\right] = 28 - 3x^2 + 2x^3$

For $x \in [0,1]$ $p_f(0,x) = \frac{1}{2}\left[V_f(0,x) + f(x) - f(0)\right] = 3x^2 - 2x^3$

$\therefore p_f(-2,x) = p_f(-2,0) + p_f(0,x) = 3x^2 - 2x^3$

$n_f(0,x) = 0$

$\therefore n_f(-2,x) = n_f(-2,0) + n_f(0,x) = 28$

For $x \in [1,2]$ $p_f(1,x) = 0$ and $n_f(1,x) = 1 - 3x^2 + 2x^3$

$\therefore\ p_f(-2,x) = p_f(-2,0) + p_f(0,1) + p_f(1,x) = 1$ and

$n_f(-2,x) = n_f(-2,0) + n_f(0,1) + n_f(1,x) = 29 - 3x^2 + 2x^3$

$$p_f(-2,x) = \begin{cases} 0, & x \in [-2,0] \\ 3x^2 - 2x^3, & x \in (0,1] \\ 1, & x \in (1,2] \end{cases}$$ and

$$n_f(-2,x) = \begin{cases} 28 - 3x^2 + 2x^3, & x \in [-2,0] \\ 0, & x \in (0,1] \\ 29 - 3x^2 + 2x^3, & x \in (1,2] \end{cases}$$

Exercise 2.6.

1. Find $V_f(0,x), p_f(0,x), n_f(0,x)$ if $f(x) = x + 2[x], x \in [0,3]$

2. Find $V_f(0,x), p_f(0,x), n_f(0,x)$ if $f(x) = [x] - x, x \in [0,2]$

2.7. Curves and Paths

Definition 2.7.1 A continuous function $f:[a,b] \to R^n$ is called a curve in R^n on [a, b]. The function f itself is called a path.

If f(a) = f(b), then f is said to be a closed curve.

If $f:[a,b] \to R^n$ is a one – one function, then f is called an arc.

Note. Different paths can trace out the same curve.

e.g. $f(t)=x(t)+iy(t)=e^{i2\pi t}$, $g(t)=x(t)+iy(t)=e^{-i2\pi t}$, $0\le t\le 1$ trace out the sane circle $x^2+y^2=1$, but the points are visited in opposite directions.

$f(t)=x(t)+iy(t)=\cos 2\pi t+i\sin 2\pi t, g(t)=x(t)+iy(t)=\cos 2\pi t-i\sin 2\pi t$

$x(t)=\cos 2\pi t, y(t)=\sin 2\pi t$ and $x(t)=\cos 2\pi t, y(t)=-\sin 2\pi t$

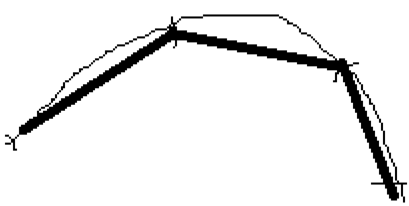
Rectifiable Paths and Arc length 2.7.2 (Rectifiable means process of finding the length of an arc of the curve)

Let $f:[a,b]\to R^n$ be a path in R^n. Let $P=\{a=x_0,x_1,x_2,........,x_n=b\}$ be partition of [a, b]. The points $f(x_0), f(x_1), f(x_2),......., f(x_n)$ are vertices of inscribed polygon.

Then the length of polygonis $\Lambda_f(P)=\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$

As we go on refinement P, the polygon approaches the range of f more and more closely, so that length of f may be defined as $\Lambda_f(a,b)=\sup_P \Lambda_f(P)$

If $\Lambda_f(a,b)<\infty$, then f is said to be a rectifiable path, otherwise it is called non – rectifiable.



E.g. $f:[0,2\pi]\to R^2$ is defined by $f(t)=x(t)+iy(t)=e^{it}$, $0\le t\le 2\pi$

$\Rightarrow x(t)=\cos t, y(t)=\sin t$

$$P_1=\left\{0,\frac{\pi}{2},\pi,\frac{3\pi}{2},2\pi\right\}$$

At t = 0, (x, y) = (1, 0), $t=\frac{\pi}{2}$, (x, y) = (0, 1)

Then length = $\sqrt{2}$ so total length = $4\sqrt{2}=5.656$

If we take partition $P_2=\left\{0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},\pi,\frac{5\pi}{4},\frac{3\pi}{2},\frac{7\pi}{4},2\pi\right\}$

At t = 0, (x, y) = (1,0), $t=\frac{\pi}{4}$, $(x,y)=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

Then length = $\sqrt{2-\sqrt{2}}=0.7655$ so total length = 8 x 0.7655 = 6.124

Actual length of arc of circle = 6.285

So we continuous refinement of partition we got actual length.

Theorem 2.7.1 Let $f_1, f_2, f_3,.........., f_n$ be real valued functions defined on [a, b] and let $f:[a,b]\to R^n$ be a vector valued function defined as $f(t)=(f_1(t), f_2(t), f_3(t),...., f_n(t))$. Then f is rectifiable if and only if each component f_k is of bounded variation on [a, b].Further if f is rectifiable, then

$$V_k(a,b)\le\Lambda_f(a,b)\le V_1(a,b)+V_2(a,b)+........+V_n(a,b)$$

Where k = 1,2,3, …..,n

Where $V_k(a,b)$ denotes the total variation of f_k on [a, b]

Proof. Let $P=\{a=t_0,t_1,t_2,........,t_n=b\}$ be a partition of [a, b].

Then

$$\left|f_k(t_i)-f_k(t_{i-1})\right|\le\sqrt{\sum_{j=1}^{n}\left[f_j(t_i)-f_j(t_{i-1})\right]^2}\le\sum_{j=1}^{n}\left|f_j(t_i)-f_j(t_{i-1})\right|$$

{ e.g. f(t) = (t, t + 1) on [1, 3]. Let P = {1, 2, 3} be partition.

$f_1(t)=t,\ f_2(t)=t+1$

$\left|f_2(t_3)-f_k(t_2)\right|=|4-3|=1,$

$$\sqrt{\sum_{j=1}^{2}\left[f_j(t_3)-f_j(t_2)\right]^2}=\sqrt{\left[f_1(t_3)-f_1(t_2)\right]^2+\left[f_2(t_3)-f_2(t_2)\right]^2}=$$

$,\ \sum_{j=1}^{2}\left|f_j(t_3)-f_j(t_2)\right|=2$ }

$$\Rightarrow\sum_{i=1}^{n}\left|f_k(t_i)-f_k(t_{i-1})\right|\le\sum_{i=1}^{n}\sqrt{\sum_{j=1}^{n}\left[f_j(t_i)-f_j(t_{i-1})\right]^2}$$

$$\le\sum_{i=1}^{n}\sum_{j=1}^{n}\left|f_j(t_i)-f_j(t_{i-1})\right|$$

$$\Rightarrow\sum_{i=1}^{n}\left|f_k(t_i)-f_k(t_{i-1})\right|\le\Lambda_f(P)\le\sum_{i=1}^{n}\sum_{j=1}^{n}\left|f_j(t_i)-f_j(t_{i-1})\right| \text{ ...... (i)}$$

From first inequality of (i), we have $\sum_{i=1}^{n}\left|f_k(t_i)-f_k(t_{i-1})\right|\le\Lambda_f(P)$

$$\Rightarrow\sup_P\sum_{i=1}^{n}\left|f_k(t_i)-f_k(t_{i-1})\right|\le\sup_P\Lambda_f(P)$$

$\Rightarrow V_k(a,b) \le \Lambda_f(a,b)$ ………………… (ii)

If f is rectifiable, then $\Lambda_f(a,b) < \infty$

$\therefore V_k(a,b) < \infty$

$\therefore$ Each component f_k is of bounded variation on [a, b]

From second inequality of (i), we have $\Lambda_f(P) \le \sum_{i=1}^{n} \sum_{j=1}^{n} \left| f_j(t_i) - f_j(t_{i-1}) \right|$

Taking supremum, we get $\Lambda_f(a,b) \le \sum_{i=1}^{n} V_j(a,b)$ ………………(iii)

If each component f_k is of bounded variation on [a, b], then $V_k(a,b) < \infty$

$\therefore \Lambda_f(a,b) < \infty$

$\Rightarrow$ f is rectifiable

Combining inequality (ii) and (iii), we get the required second part.

Theorem 2.7.2 If $f:[a,b] \to R^n$ is a curve such that f' is continuous on [a, b], then f is rectifiable and $\Lambda_f(a,b) = \int_a^b |f'(t)| dt$

Proof. Let $P = \{a = t_0, t_1, t_2, \ldots\ldots, t_n = b\}$ be a partition of [a, b].

Then $\left| f(t_i) - f(t_{i-1}) \right| = \left| \int_{t_{i-1}}^{t_i} f'(t) dt \right| \le \int_{t_{i-1}}^{t_i} |f'(t)| dt$

Putting $i = 1,2,3,\ldots.n$ and adding, we get $\sum_{i=1}^{n} \left| f(t_i) - f(t_{i-1}) \right| \le \int_a^b |f'(t)| dt$

$$\Rightarrow \Lambda_f(P) \le \int_a^b |f'(t)|dt \Rightarrow \sup_P \Lambda_f(P) \le \int_a^b |f'(t)|dt$$

$$\Rightarrow \Lambda_f(a,b) \le \int_a^b |f'(t)|dt \quad \text{.....................} \text{(i)}$$

T.P $\Lambda_f(a,b) \ge \int_a^b |f'(t)|dt$

Since f' is continuous on [a, b], therefore f' is uniformly continuous on [a, b].

$\therefore$ For given $\varepsilon > 0$, there exist $\delta > 0$ such that $|f'(s) - f'(t)| < \varepsilon$ for $|s-t| < \delta$

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition of [a, b] with $\|P\| < \delta$

Then $|f'(t) - f'(x_i)| < \varepsilon$ for $x_{i-1} \le t \le x_i$

$\therefore |f'(t)| - |f'(x_i)| \le |f'(t) - f'(x_i)| < \varepsilon \qquad [\because |a-b| \ge |a| - |b|]$

$\therefore |f'(t)| < \varepsilon + |f'(x_i)|$

$$\Rightarrow \int_{x_{i-1}}^{x_i} |f'(t)|dt \le |f'(x_i)|\Delta x_i + \varepsilon\Delta x_i = \left|\int_{x_{i-1}}^{x_i} [f'(t) + f'(x_i) - f'(t)]dt\right| + \varepsilon\Delta x_i$$

$$\leq \left| \int_{x_{i-1}}^{x_i} f'(t)dt \right| + \left| \int_{x_{i-1}}^{x_i} [f'(x_i) - f'(t)]dt \right| + \varepsilon \Delta x_i$$

$$\leq \left| f(x_i) - f(x_{i-1}) \right| + \left| \int_{x_{i-1}}^{x_i} \varepsilon dt \right| + \varepsilon \Delta x_i \leq \left| f(x_i) - f(x_{i-1}) \right| + \varepsilon \left| x_i - x_{i-1} \right| + \varepsilon \Delta x_i$$

$$= \left| f(x_i) - f(x_{i-1}) \right| + 2\varepsilon \Delta x_i$$

Putting $i = 1,2,3,....,n$ and adding, we have

$$\int_a^b |f'(t)|dt \leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + 2\varepsilon(b-a) \leq \Lambda_f(P) + 2\varepsilon(b-a)$$
$$= \sup_P \Lambda_f(P) + 2\varepsilon(b-a)$$

$$\int_a^b |f'(t)|dt \leq \Lambda_f(a,b) + 2\varepsilon(b-a)$$

But ε is arbitrary

$$\Rightarrow \int_a^b |f'(t)|dt \leq \Lambda_f(a,b) \qquad \text{..................} \text{(ii)}$$

From (i) and (ii)

$$\Lambda_f(a,b) = \int_a^b |f'(t)|dt$$

Example 2.7.1 If $f : [0,2\pi] \to R^2$ is defined by $f(t) = (a\cos t, b\sin t)$. Show that f is rectifiable and find its length.

Sol. $f(t)=(a\cos t, b\sin t) \Rightarrow f'(t)=(-a\sin t, b\cos t)$

This is clearly continuous in $[0,2\pi]$

$\therefore$ f is rectifiable

$$\Lambda_f(0,2\pi)=\int_0^{2\pi}|f'(t)|dt=\int_0^{2\pi}\sqrt{(-a\sin t)^2+(b\cos t)^2}\,dt=\int_0^{2\pi}a\,dt=2\pi a$$

Example 2.7.2 If $f:[0,1]\to R^3$ is defined by $f(t)=(at^2, 2at, at)$. Find the length of curve.

Sol. $f'(t)=(2at, 2a, a)$

$$\therefore \Lambda_f(0,1)=\int_0^1\sqrt{4a^2t^2+4a^2+a^2}\,dt=a\int_0^1\sqrt{4t^2+5}\,dt=2a\int_0^1\sqrt{t^2+\left(\frac{\sqrt{5}}{2}\right)^2}\,dt$$

$$=2a\left\{\frac{t}{2}\sqrt{t^2+\frac{5}{4}}+\frac{5}{8}\log\left|t+\sqrt{t^2+\frac{5}{4}}\right|\right\}_0^1=2a\left\{\frac{1}{2}\times\frac{3}{2}+\frac{5}{8}\log\frac{5}{2}-\frac{5}{8}\log\sqrt{\frac{5}{4}}\right\}$$

$$=2a\left\{\frac{1}{2}\times\frac{3}{2}+\frac{5}{8}\log\frac{5}{2}-\frac{5}{16}\log\frac{5}{4}\right\}=\frac{a}{8}\left\{12+10\log\frac{5}{2}-5\log\frac{5}{4}\right\}$$

$$=\frac{a}{8}\{12+10\log 5-5\log 5\}=\frac{a}{8}\{12+5\log 5\}$$

Exercise 2.7

1. Show that the length of the curve

(i) x = a cost, y = a sin t, z = a t, $0 \le t \le 2\pi$ is $2\sqrt{2}a\pi$

(ii)x = 2 t – 1, y = t + 1, z = t – 2, $0 \le t \le 3$ is $3\sqrt{6}$

(iii) $x = a(\theta - \sin\theta), y = a(1-\cos\theta), z = a\theta, -\pi \le \theta \le \pi$ is

$$\int_0^{\pi} 2a\sqrt{3-2\cos\theta}\,d\theta$$

2.8. Additive and continuity properties of arc length.

Theorem 2.8.1 If $c \in (a,b)$, then $\Lambda_f(a,b) = \Lambda_f(a,c) + \Lambda_f(c,b)$

Proof. Introducing the point c to a partition P of [a, b], we get a partition P_1 of [a, c] and a partition P_2 of [c, b] such that $\Lambda_f(P) \le \Lambda_f(P_1) + \Lambda_f(P_2)$

$\Rightarrow \Lambda_f(a,b) \le \Lambda_f(a,c) + \Lambda_f(c,b)$ .... (i)

Conversely. We consider a partition P_1 of [a, c] and a partition P_2 of [c, b], then $P = P_1 \cup P_2$ is a partition of [a, b] and

$\Lambda_f(P_1) + \Lambda_f(P_2) = \Lambda_f(P) \le \Lambda_f(a,b)$

$\Rightarrow \Lambda_f(a,c) + \Lambda_f(c,b) \le \Lambda_f(a,b)$.......... (ii)

From (i) and (ii), we get $\Lambda_f(a,b) = \Lambda_f(a,c) + \Lambda_f(c,b)$

Theorem 2.8.2. Let $f:[a,b] \to R^n$ be a rectifiable path. For $x \in [a,b]$, let $s(x) = \Lambda_f(a,x)$ and s(a) = 0. Then

(i) The function s is increasing and continuous on [a, b]

(ii) If there is no subinterval of [a, b] on which f is constant, then s is strictly increasing on [a, b].

Proof: (i) Let $a \le x < y \le b$

T.P. $s(x) \le s(y)$

Then $s(y) = \Lambda_f(a, y) = \Lambda_f(a, x) + \Lambda_f(x, y)$

$\Rightarrow s(y) = s(x) + \Lambda_f(x, y)$

$\Rightarrow s(y) - s(x) = \Lambda_f(x, y) \ge 0$ .... (a)

$\Rightarrow s(y) \ge s(x)$

Hence s is increasing on [a, b].

Next we prove s is continuous on [a, b]

For this let $a \le x < y \le b$

Then $0 \le s(y) - s(x) = \Lambda_f(x, y) \le \sum_{k=1}^{n} V_k(x, y)$

If $y \to x$, then $V_k(x, y) \to 0 \ \forall k$

$\therefore \ s(x) = s(x^+)$

Similarly, we can show that $s(x) = s(x^-)$

Hence s is continuous on [a,b]

(ii) we are given that f is not constant on any subinterval of [a, b]

$\therefore \Lambda_f(x,y) \neq 0$

$\therefore \Lambda_f(x,y) > 0$

From (a), $s(y) - s(x) = \Lambda_f(x,y) > 0 \Rightarrow$ s(y) > s(x)

Hence s is strictly increasing on [a,b].

2.9. Equivalence of Paths and change of Parameter.

Definition 2.9.1. Let $f:[a,b] \to R^n$ be a path. Let $u:[c,d] \to [a,b]$ be a real valued continuous function which is strictly monotonic on [c, d] having range [a, b]. Then the composite function $g = f \circ u$ defined by $g(t) = f(u(t)), c \le t \le d$ is a path having the same graph as f.

The paths f and g related as above are said to be equivalent paths.

The function u is said to define a change of parameter.

Let C denote the common graph of two equivalent paths f and g. If u is strictly increasing, we say f and g trace out C in the same direction. If u is strictly decreasing, we say f and g trace out C in opposite directions. In this first case, u is said to be orientation – preserving. In second case u is said to be orientation reversing.

Theorem 2.9.1 Let $f:[a,b] \to R^n$ and $g:[c,d] \to R^n$ be two paths in R^n, each of which is injective on its domain. Then f and g are equivalent paths if and only if they have the same graph.

Sol. Let f and g are equivalent paths then clearly f and g have the same graph.

Conversely let f and g have the same graph.

Since f is injective and continuous on $[a,b]$, then f^{-1} exists and is continuous on its graph.

Define $u(t) = f^{-1}[g(t)], t \in [c,d]$. Then u is continuous on $[c,d]$ and $g(t) = f[u(t)]$

$\therefore$ u is strictly monotonic and hence f and g are equivalent paths.

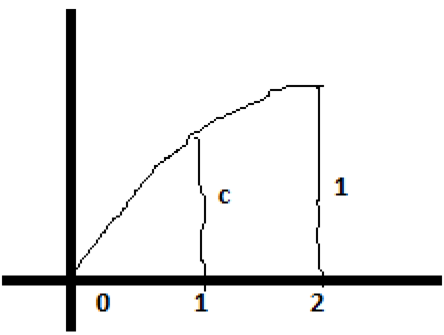
Example 2.9.1 Let f and g be complex valued functions defined as follows:

$f(t) = e^{2\pi it}, t \in [0,1]$ and $g(t) = e^{2\pi it}, t \in [0,2]$. Prove that f and g have the same graph but are not equivalent according to definition.

Sol. Since $f(t) = e^{2\pi it}, t \in [0,1]$ and $g(t) = e^{2\pi it}, t \in [0,2]$ represented the circle of unit disk, so f and g have the same graph.

If f and g are equivalent, then there is an monotonic function $u:[0,2] \to [0,1]$ such that $g(t) = f(u(t))$

$f(u(t)) = \cos 2\pi(u(t)) + i\sin 2\pi(u(t)) = g(t) = \cos 2\pi t + i\sin 2\pi t$



In particular, u(1) = $c \in (0,1)$

$f(u(1)) = g(1) = 1 \Rightarrow \cos 2\pi c + i\sin 2\pi c = 1 \Rightarrow$ c is integer, a contradiction

$\therefore$ f and g are not equivalent,

Chapter3.

The Riemann –Stieltjes Integral

3.1. Definition

Partition 3.1.1. By a partition P of a closed interval [a, b] we mean a finite set $\{a = x_0, x_1, x_2, ..., x_n = b\}$ of real numbers such that
$a = x_0 < x_1 < x_2 < .... < x_n = b$

Note. $\Delta x_i = x_i - x_{i-1}$ for $i = 1,2,.....,n$ Δx_i is called the length of the segment $[x_{i-1}, x_i]$. The greatest of the length of the segments of partition P will be denoted by $\|P\|$ or $\mu(P)$ and will be closed norm of the partition.

Refinement 3.1.2. A partition P^* will be called refinement of another partition P if and only if $P \subset P^*$ i.e. every element of P is in P^*. It is clear that $\|P^*\| \le \|P\|$

Common Refinement 3.1.3 Give two partition P_1 and P_2, we say P^* is their common refinement if $P^* = P_1 \cup P_2$

Lower and upper Riemann –stieltjes sums 3.1.4 Let f be a bounded real valued function defined on [a, b] and α be monotonically non – decreasing real valued function on [a, b] corresponding to each partition. Let

P = $\{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition on [a, b]. we write
$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ and $\Delta\alpha_i \ge 0$

$\sum \Delta\alpha_i = \alpha(b) - \alpha(a)$

Let $m = \inf .\{f(x) : a \le x \le b\}, M = \sup .\{f(x) : a \le x \le b\}$ and
$m_i = \inf .\{f(x) : x_{i-1} \le x \le x_i\}, M_i = \sup .\{f(x) : x_{i-1} \le x \le x_i\}$ We define

lower and upper R – S sums of f w.r.t. α as

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \text{ and } U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

Note If $\alpha(x) = x$, then above lower and upper sum are called Riemann lower and upper sums.

Theorem 3.1.1 Prove the following with P partition and P^* is its refinement

$(i) L(P, f, \alpha) \le U(P, f, \alpha)$ (ii) $L(P, f, \alpha) \le L(P^*, f, \alpha)$

(iii) $U(P^*, f, \alpha) \le U(P, f, \alpha)$

Proof: (i) Let P = $\{a = x_0, x_1, x_2, ..., x_n = b\}$ be a partition on [a, b].

Let $m_i = \inf.\{f(x) : x_{i-1} \le x \le x_i\}, M_i = \sup.\{f(x) : x_{i-1} \le x \le x_i\}$

Clearly $m_i \le M_i \Rightarrow m_i \Delta\alpha_i \le M_i \Delta\alpha_i \Rightarrow \sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i$

$\Rightarrow L(P, f, \alpha) \le U(P, f, \alpha)$

(ii) Let P^* has one more point of partition of P.

Let $P = \{a = x_0, x_1, ..., x_{r-1}, x_r, ..., x_n = b\}$ and

$P^* = \{a = x_0, x_1, ..., x_{r-1}, t, x_r, ..., x_n = b\}$ be partition of [a, b] and P^* is refinement of P.

Let $m_i = \inf.\{f(x) : x_{r-1} \le x \le x_r\}, M_i = \sup.\{f(x) : x_{r-1} \le x \le x_r\}$

$m'_r = \inf\{f(x) : x_{r-1} \le x \le t\}, M'_r = \inf\{f(x) : x_{r-1} \le x \le t\}$

$m''_r = \inf\{f(x) : t \le x \le x_r\}, M''_r = \inf\{f(x) : t \le x \le x_r\}$

Clearly $m_i \le m'_r, m_i \le m''_r, M'_r \le M_i, M''_r \le M_i$

$$L(P^*, f, \alpha) = \sum_{i=1}^{r-1} m_i \Delta\alpha_i + m'_r \{\alpha(t) - \alpha(x_{r-1})\} + m''_r \{\alpha(x_r) - \alpha(t)\} + \sum_{i=r+1}^{n} m_i \Delta\alpha$$

$$\geq \sum_{i=1}^{r-1} m_i \Delta\alpha_i + m_i\{\alpha(t) - \alpha(x_{r-1})\} + m_i\{\alpha(x_r) - \alpha(t)\} + \sum_{i=r+1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{r-1} m_i \Delta\alpha_i + m_i\{\alpha(x_r) - \alpha(x_{r-1})\} + \sum_{i=r+1}^{n} m_i \Delta\alpha_i = L(P, f, \alpha)$$

(iii)

$$U(P^*, f, \alpha) = \sum_{i=1}^{r-1} M_i \Delta\alpha_i + M'_r\{\alpha(t) - \alpha(x_{r-1})\} + M''_r\{\alpha(x_r) - \alpha(t)\} + \sum_{i=r+1}^{n} M_i$$

$$\leq \sum_{i=1}^{r-1} M_i \Delta\alpha_i + M_i\{\alpha(t) - \alpha(x_{r-1})\} + M_i\{\alpha(x_r) - \alpha(t)\} + \sum_{i=r+1}^{n} M_i \Delta\alpha_i$$

$$= \sum_{i=1}^{r-1} M_i \Delta\alpha_i + M_i\{\alpha(x_r) - \alpha(x_{r-1})\} + \sum_{i=r+1}^{n} M_i \Delta\alpha_i = \sum_{i=1}^{n} M_i \Delta\alpha_i =$$

$$U(P, f, \alpha)$$

Note. If P_1 and P_2 are two partition of [a, b] and $P = P_1 \cup P_2$ the common refinement. It is clear that $L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq U(P_2, f, \alpha)$

3.2. Definition

Lower and upper Riemann –Stieltjes integrals Let f be a bounded real valued function defined on [a, b] and α be monotonically non – decreasing real valued function on [a, b] corresponding to each partition. Then **Lower Riemann –**

Stieltjesintegral = $\int_{\underline{a}}^{b} f d\alpha = Sup L(P, f, \alpha)$ and

upper Riemann –Stieltjesintegral = $\int_{a}^{\overline{b}} f d\alpha = \inf\ U(P, f, \alpha)$

When **Lower and upper Riemann –Stieltjesintegrals** are equal, then their common value is called **Riemann –Stieltjesintegral** and is written as

$\int_a^b f d\alpha = \int_a^b f(x) d\alpha(x)$ and we say f is **Riemann –Stieltjes integral** with respect to α on [a, b] in symbols, we write as $f \in R(\alpha)$ on [a, b] or $f \in R(\alpha,[a,b])$

Students easy understand the above concept (Theorem 3.1.1 and definition 3.2) by given below Example

Example3.2.1. If $f(x) = \cos x$ and $\alpha(x) = \sin x$. Find $L(P, f, \alpha)$ and $U(P, f, \alpha)$ on $\left[0, \frac{\pi}{2}\right]$. Also verify theorem 3.1.1 and definition 3.2.

Sol. Let $P = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ be a partition of $\left[0, \frac{\pi}{2}\right]$, clearly f(x) is decreasing and $\alpha(x)$ is increasing function.

For $I_1 = \left[0, \frac{\pi}{4}\right]$ then $m_1 = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, M_1 = \cos 0 = 1$,

$$\Delta\alpha_1 = \alpha\left(\frac{\pi}{4}\right) - \alpha(0) = \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

For $I_2 = \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ then $m_2 = \cos\frac{\pi}{2} = 0, M_2 = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$,

$$\Delta\alpha_2 = \alpha\left(\frac{\pi}{2}\right) - \alpha\left(\frac{\pi}{4}\right) = 1 - \frac{1}{\sqrt{2}}$$

$$L(P, f, \alpha) = m_1\Delta\alpha_1 + m_2\Delta\alpha_2 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + 0 = \frac{1}{2} = 0.5$$

$$U(P, f, \alpha) = M_1\Delta\alpha_1 + M_2\Delta\alpha_2 = 1 \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{2} - \frac{1}{2} = 0.914$$

Take new partition P^* where P^* is refinement of P.

Let $P^* = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$ be a partition on $\left[0, \frac{\pi}{2}\right]$

For $I_1 = \left[0, \frac{\pi}{6}\right]$ then $m1 = \frac{\sqrt{3}}{2}, M1 = 1, \Delta\alpha 1 = \frac{1}{2} - 0 = \frac{1}{2}$

For $I_2 = \left[\frac{\pi}{6}, \frac{\pi}{4}\right]$ then $m2 = \frac{1}{\sqrt{2}}, M2 = \frac{\sqrt{3}}{2}, \Delta\alpha 2 = \frac{1}{\sqrt{2}} - \frac{1}{2}$

For $I_3 = \left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ then $m3 = \frac{1}{2}, M3 = \frac{1}{\sqrt{2}}, \Delta\alpha 3 = \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}$

For $I_4 = \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ then $m4 = 0, M4 = \frac{1}{2}, \Delta\alpha 4 = 1 - \frac{\sqrt{3}}{2}$

$$L\left(P^*, f, \alpha\right) = m_1\Delta\alpha_1 + m_2\Delta\alpha_2 + m_3\Delta\alpha_3 + m_4\Delta\alpha_4$$

$$= \frac{\sqrt{3}}{2}.\frac{1}{2} + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right) = 0.659$$

$$U\left(P^*, f, \alpha\right) = M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + M_3\Delta\alpha_3 + M_4\Delta\alpha_4$$

$$= 1 \times \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) = 0.858$$

$\therefore$ (i) $L(P, f, \alpha) \le U(P, f, \alpha)$ (ii) $L(P, f, \alpha) \le L\left(P^*, f, \alpha\right)$ (iii) $U\left(P^*, f, \alpha\right) \le U(P, f, \alpha)$

Also $\int_0^{\frac{\pi}{2}} f(x)d\alpha(x) = \int_0^{\frac{\pi}{2}} \cos x.\cos x dx = \int_0^{\frac{\pi}{2}} \cos^2 x dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4} = 0.786$$

$L(P, f, \alpha) = 0.5,\ L(P^*, f, \alpha) = 0.659$, ... and $U(P, f, \alpha) = 0.914$, $U(P^*, f, \alpha) = 0.858$, ...

Continuously we take new partition P^{**} is refinement of P^* increasing the value of**Lower Riemann –Stieltjessum** and decreasing the value of **Upper Riemann – Stieltjessum.**

We get $\underline{\int_0^{\frac{\pi}{2}}} f(x)d\alpha(x) = \sup .L(P, f, \alpha)$ and $\overline{\int_0^{\frac{\pi}{2}}} f(x)d\alpha(x) = \inf .U(P, f, \alpha)$

Theorem 3.2.1 Let f be a bounded function and α be non – decreasing function on [a, b], then $\underline{\int_a^b} fd\alpha \le \overline{\int_a^b} fd\alpha$

Proof: Let P_1 and P_2 are two partition of [a, b] and $P = P_1 \cup P_2$ the common refinement. We get $L(P_1, f, \alpha) \le U(P_2, f, \alpha)$

$U(P_2, f, \alpha)$ isupper bound of $L(P_1, f, \alpha)$

But $\underline{\int_a^b} f d\alpha = Sup L(P, f, \alpha)$

we know that $l.u.b. \leq u.b$

$\therefore \underline{\int_a^b} f d\alpha \leq U(P_2, f, \alpha)$

$\underline{\int_a^b} f d\alpha$ is lower bound of $U(P_2, f, \alpha)$

But $\overline{\int_a^b} f d\alpha = \inf\ U(P, f, \alpha)$

We know that $g.l.b. \geq l.b.$

$\therefore \overline{\int_a^b} f d\alpha \geq \underline{\int_a^b} f d\alpha$

$\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$.

Cor. For any partition P of [a, b],

$$L(P, f, \alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(P, f, \alpha)$$

Theorem 3.2.2 Let f be bounded and α a non -decreasing function on [a, b], then $f \in R(\alpha)$ if and only if for every $\varepsilon > 0$, $\exists$ a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Proof: Let $f \in R(\alpha)$

$$\therefore \underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha \quad \text{.................. (i)}$$

T.P. $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Since $\underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha)$ over all P

there exist a partition P_1 such that

$$L(P_1, f, \alpha) > \underline{\int_a^b} f d\alpha - \frac{\varepsilon}{2} \quad \text{.................. (ii)}$$

Again $\overline{\int_a^b} f d\alpha = Inf U(P, f, \alpha)$ over all P

there exist a partition P_2 such that

$U(P_2, f, \alpha) < \overline{\int_a^b} f d\alpha + \frac{\varepsilon}{2}$ .... (iii) Let

$P = P_1 \cup P_2$ so P is common refinement

$\therefore\ L(P_1, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_2, f, \alpha)$

$U(P, f, \alpha) \le U(P_2, f, \alpha) < \overline{\int_a^b} f d\alpha + \frac{\varepsilon}{2}$ from (iii)

From (i) we get $U(P, f, \alpha) < \int_a^b f d\alpha + \frac{\varepsilon}{2}$.................. (iv)

Also $L(P, f, \alpha) \ge L(P_1, f, \alpha) > \underline{\int_a^b} f d\alpha - \frac{\varepsilon}{2}$ from (ii)

from (i) we get $L(P, f, \alpha) > \int_a^b f d\alpha - \frac{\varepsilon}{2}$

$\Rightarrow -L(P, f, \alpha) < -\int_a^b f d\alpha + \frac{\varepsilon}{2}$ .... (v)

Adding (iv) and (v) we get $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Conversely Let $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Now we prove that $f \in R(\alpha)$ over [a, b]

We known that $L(P, f, \alpha) \leq \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \leq U(P, f, \alpha)$

$$0 \leq \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha \leq U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

$$\therefore \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha < \varepsilon$$

Since ε is arbitrary, we have $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha \Rightarrow f \in R(\alpha)$ over [a, b]

Theorem 3.2.3 Let $f : [a, b] \to R$ be a bounded function and let α be a monotonic increasing function on $[a, b]$.If f is continuous on the [a, b] then $f \in R(\alpha)$on [a, b].

Proof: Let $\varepsilon > 0$ be given. Now f, being continuous on [a, b], is uniformly continuous on [a, b].

$\therefore \exists\ \delta > 0$ Such that

$|f(x) - f(y)| < \dfrac{\varepsilon}{\alpha(b) - \alpha(a) + 1}, \forall x, y \in [a, b]$Whenever $|x - y| < \delta$ …… (i)

Let $P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$ be a partition of [a, b] with $\|P\| < \delta$

Since a continuous function on closed domain [a, b] is bounded and attains its bounds in [a, b], therefore there exist $c_i, d_i \in [x_{i-1}, x_i]$ such that $f(c_i) = m_i$

and $f(d_i) = M_i$ where

$m_i = \inf\{f(x) : x_{i-1} < x < x_i\}, M_i = \sup\{f(x) : x_{i-1} < x < x_i\}$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= \sum_{i=1}^{n} (f(d_i) - f(c_i))\Delta\alpha_i$$

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon}{\alpha(b) - \alpha(a) + 1} \sum_{i=1}^{n} \Delta\alpha_i$$ From (i)

$$= \frac{\varepsilon}{\alpha(b) - \alpha(a) + 1} \times \alpha(b) - \alpha(a) < \varepsilon$$

$\therefore U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

$\therefore f \in R(\alpha)$ on[a, b].

Theorem 3.2.4 If f is monotonic on [a, b] and if α is continuous on [a, b] then $f \in R(\alpha)$ on [a, b].

Proof: Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be a partition of [a, b].

α is continuous on [a, b]

$\therefore \alpha$ takes all values between $\alpha(a)$ and $\alpha(b)$

By intermediate value theorem

Choose $\Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$ where $i = 1, 2, ..., n$

Since f is monotonic

$\therefore$ either f is monotonic increasing or monotonic decreasing.

Let us suppose that f is monotonic increasing.

Let M_i, m_i be the bound of f on $[x_{i-1}, x_i]$

Then $M_i = f(x_i), m_i = f(x_{i-1})$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})]$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \times [f(b) - f(a)]$$

Since $[\alpha(b) - \alpha(a)] \times [f(b) - f(a)]$ is a fixed constant.

By taking n sufficiently large

$\dfrac{[\alpha(b) - \alpha(a)][f(b) - f(a)]}{n}$ Can be made less than ε

Where $\varepsilon > 0$ is arbitrary $\therefore$

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

$\therefore\ f \in R(\alpha)$ on [a, b].

Theorem 3.2.5 Let f be bounded on [a, b] having finitely many points of discontinuity in [a, b] and let α be monotonically increasing function which is continuous at all those points where f is discontinuous.
Then $f \in R(\alpha)$ on [a, b].

Proof: Let $\{c_1, c_2, \ldots\ldots, c_p\}$ be the set of finitely many points at which f is discontinuous such that $c_1 < c_2 < \ldots\ldots < c_p$. We can enclose these p points in p non – overlapping intervals

$[a_1, b_1], [a_2, b_2], \ldots\ldots\ldots, [a_p, b_p]$ ………………. (i)

$$\sum_{i=1}^{p}[\alpha(b_i) - \alpha(a_i)] < \frac{\varepsilon}{2(M - m + 1)}$$

Where M and m are bounds of f on [a, b] and it is possible as α is continuous at $c_1, c_2, \ldots\ldots, c_p$.

Now f is continuous in each of p + 1 sub – intervals

$[a, a_1], [b_1, a_2], \ldots\ldots\ldots, [b_p, b]$ .... (ii)

$\therefore$ There exist partitions $P_1, P_2, \ldots\ldots, P_{p+1}$ of $[a, a_1], [b_1, a_2], \ldots\ldots\ldots, [b_p, b]$

respectively such that $U(P_i, f, \alpha) - L(P_i, f, \alpha) < \dfrac{\varepsilon}{2(p+1)}$

Let $P = \bigcup_{i=1}^{p+1} P_i$

$\therefore$ contribution of the subintervals in (ii)

$U(P, f, \alpha) - L(P, f, \alpha) < \dfrac{\varepsilon}{2)p+1)}(p+1) = \dfrac{\varepsilon}{2}$ ....(iii)

Since each of the sub – intervals in (i) is subset of [a, b], therefore oscillation of f in any of these sun – intervals $\leq M - m$.

$\therefore U(P, f, \alpha) - L(P, f, \alpha) \leq \sum_{i=1}^{p} (M - m)[\alpha(b_i) - \alpha(a_i)]$

$= (M - m) \sum_{i=1}^{p} [\alpha(b_i) - \alpha(a_i)]$

$< (M - m) \dfrac{\varepsilon}{2(M - m + 1)} < \dfrac{\varepsilon}{2}$ .... (iv)

From (iii) and (iv) we get $U(P, f, \alpha) - L(P, f, \alpha) < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$

$\therefore\ f \in R(\alpha)$ on [a, b].

Theorem 3.2.6 If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on [a, b] then $f_1 + f_2 \in R(\alpha)$ on [a, b] and $\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$

Proof: Since $f_1 \in R(\alpha)$

$\therefore$ There exist a partition P_1 of [a, b] such that

$U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\varepsilon}{2}$ .... (i)

Since $f_2 \in R(\alpha)$

$\therefore$ There exist a partition P_2 of [a, b] such that

$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\varepsilon}{2}$ .... (ii)

Let $P = P_1 \cup P_2$ be a common refinement.

$\therefore$ (i) And (ii) will hold for P also

$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \frac{\varepsilon}{2}$ .... (iii)

And $U(P, f_2, \alpha) - L(P, f_2, \alpha) < \frac{\varepsilon}{2}$ .... (iv)

Adding (iii) and (iv)

$U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha) < \varepsilon$ .... (v)

Let $f = f_1 + f_2$

we know that $Sup(f_1 + f_2) \leq Sup(f_1) + Sup(f_2)$ and

$Inf(f_1 + f_2) \geq Inf(f_1) + Inf(f_2)$

$\therefore U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$ and

$L(P, f, \alpha) \geq L(P, f_1, \alpha) + L(P, f_2, \alpha)$

$U(P, f, \alpha) - L(P, f, \alpha)$

$\leq U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha) < \varepsilon$ From (v)

$\therefore\ f = f_1 + f_2 \in R(\alpha)$

Also $U(P_1, f_1, \alpha) < \overline{\int_a^b} f_1 d\alpha + \frac{\varepsilon}{2} = \int_a^b f_1 d\alpha + \frac{\varepsilon}{2}$ and

$U(P_2, f_2, \alpha) < \overline{\int_a^b} f_2 d\alpha + \frac{\varepsilon}{2} = \int_a^b f_2 d\alpha + \frac{\varepsilon}{2}$

Let $P = P_1 \cup P_2$ be a common refinement.

$U(P, f_1, \alpha) < \int_a^b f_1 d\alpha + \frac{\varepsilon}{2}$ and $U(P, f_2, \alpha) < \int_a^b f_2 d\alpha + \frac{\varepsilon}{2}$

Also $U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha) < \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + \varepsilon$

But $\int_a^b f d\alpha \leq U(P, f, \alpha)$

$\therefore\ \int_a^b f d\alpha < \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + \varepsilon$

Since ε is arbitrary

$\therefore\ \int_a^b f d\alpha \leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$ ....(vi)

Similarly we get $\int_a^b f d\alpha \geq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$ .... (vii)

From (vi) and (vii) we get

$$\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

Theorem 3.2.7 If $f \in R(\alpha)$ on [a, b] then $cf \in R(\alpha)$ and

$\int_a^b cf d\alpha = c\int_a^b f d\alpha$

Proof: If c = 0 then the result is obvious.

Case 1. if c > 0 then Sup c f = c Sup f and Inf c f = c Inf f

$\therefore$ for any partition P of [a, b] we have

U(P, c f, α) = c U(P, f, α) and L(P, c f, α) = c L(P, f, α)

U(P, c f, α) - L(P, c f, α) = c U(P, f, α) - c L(P, f, α)

= c {U(P, f, α) - L(P, f, α)} < c . $\frac{\varepsilon}{c} = \varepsilon$

$\Rightarrow cf \in R(\alpha)$

Also $\overline{\int_a^b} cf\, d\alpha = c \overline{\int_a^b} f\, d\alpha$

$\Rightarrow \int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$

Case 2. if c < 0. Then Sup c f = c Inf f and Inf c f = c Sup f
$\therefore$ for any partition P of [a, b] we have

U(P, c f, α) = c L(P, f, α) and L(P, c f, α) = c U(P, f, α)

U(P, c f, α) - L(P, c f, α) = c L (P, f, α) - c U(P, f, α)

= - c {U(P, f, α) - L(P, f, α)} < (-c) . $\frac{\varepsilon}{-c} = \varepsilon$

$\Rightarrow cf \in R(\alpha)$

Also $\underline{\int_a^b} cf\, d\alpha = c \underline{\int_a^b} f\, d\alpha$

$\Rightarrow \int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$

3.3.Another Definition of Integrability.

Let f be a bounded real valued function on [a, b] and let α be monotonic increasing. Let $P=\{a=x_0, x_1, x_2, \ldots\ldots, x_n=b\}$ be a partition of [a, b]. Choose $t_1, t_2, \ldots\ldots, t_n$ such that $x_{i-1} \le t_i \le x_i$, $i=1,2,\ldots,n$ and form the sum $S(P, f, \alpha)=\sum\limits_{i=1}^{n} f(t_i)\Delta\alpha_i$ (This is called sum function) If there is a number I for which to every $\varepsilon>0$, $\exists$ a positive number δ such that $|S(P, f, \alpha)-I|<\varepsilon$ whenever $\mu(P)<\delta$ i.e. $\underset{\mu(P)\to 0}{Lt} S(P, f, \alpha)=I$

Theorem 3.3.1 If $\underset{\mu(P)\to 0}{Lt} S(P, f, \alpha)$ exists, then $f \in R(\alpha)$ and

$$\underset{\mu(P)\to 0}{Lt} S(P, f, \alpha)=\int_a^b f d\alpha$$

Proof: Let $\underset{\mu(P)\to 0}{Lt} S(P, f, \alpha)=I$

$\therefore$ given $\varepsilon>0$, $\exists$ a positive number δ such that

$|S(P, f, \alpha)-I|<\frac{\varepsilon}{2}$ Whenever $\mu(P)<\delta$

$$I-\frac{\varepsilon}{2}<S(P,f,\alpha)<I+\frac{\varepsilon}{2} \text{ .... (i)}$$

Let $L(P,f,\alpha)$ and $U(P,f,\alpha)$ be the g.l.b. and l.u.b. resp. of the sums $S(P,f,\alpha)$ so obtained.

Then (i) give

$$I-\frac{\varepsilon}{2}<L(P,f,\alpha)\le U(P,f,\alpha)<I+\frac{\varepsilon}{2} \text{ .... (ii)}$$

$$\Rightarrow U(P,f,\alpha)<I+\frac{\varepsilon}{2} \text{ and } L(P,f,\alpha)>I-\frac{\varepsilon}{2}$$

i.e. $-L(P,f,\alpha)<-I+\frac{\varepsilon}{2}$

$$U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$$

$$\therefore\ f\in R(\alpha)$$

Also $L(P,f,\alpha)\le\int_a^b f d\alpha\le U(P,f,\alpha)$

From (ii) we have $I-\frac{\varepsilon}{2}<\int_a^b f d\alpha<I+\frac{\varepsilon}{2}$

$$\Rightarrow\left|\int_a^b f d\alpha-I\right|<\frac{\varepsilon}{2}<\varepsilon$$

$$\Rightarrow\int_a^b f d\alpha=I=\underset{\mu(P)\to 0}{Lt}\ S(P,f,\alpha)$$

3.4.Properties of Riemann – Stieltjes Integral

Theorem 3.4.1 If $f, g \in R(\alpha)$ on [a, b], then $Af + Bg \in R(\alpha)$ on [a, b] and $\int_a^b (Af + Bg)d\alpha = A\int_a^b f\,d\alpha + B\int_a^b g\,d\alpha$ where A and B are real constants.

Proof: For a given partition P of [a, b],

Let $\phi = Af + Bg$, then

$$S(P,\phi,\alpha) = \sum_{i=1}^{n} \phi(t_i)\Delta\alpha_i = \sum_{i=1}^{n} (Af + Bg)(t_i)\Delta\alpha_i = A\sum_{i=1}^{n} f(t_i)\Delta\alpha_i + B\sum_{i=1}^{n} g(t_i)\Delta$$

$$\therefore\ S(P,\phi,\alpha) = AS(P, f,\alpha) + BS(P, g,\alpha)$$

Let $\varepsilon > 0$ be given

since $f, g \in R(\alpha)$, there exist partitions P_f and P_g such that

$$\left| S(P, f,\alpha) - \int_a^b f\,d\alpha \right| < \frac{\varepsilon}{2|A|} \text{ and } \left| S(P, g,\alpha) - \int_a^b g\,d\alpha \right| < \frac{\varepsilon}{2|B|}$$

Where $P_f \subseteq P, P_g \subseteq P$

If $P' = P_f \cup P_g$ then for any partition P finer than P'

$$\left|S(P,\phi,\alpha)-A\int_a^b f\,d\alpha - B\int_a^b g\,d\alpha\right| = \left|AS(P,f,\alpha)+BS(P,g,\alpha)-A\int_a^b f\,d\alpha - B\int_a^b g\,d\alpha\right|$$

$$=\left|AS(P,f,\alpha)-A\int_a^b f\,d\alpha + BS(P,g,\alpha)-B\int_a^b g\,d\alpha\right|$$

$$\leq |A|\left|S(P,f,\alpha)-A\int_a^b f\,d\alpha\right| + |B|\left|S(P,g,\alpha)-\int_a^b g\,d\alpha\right|$$

$$< |A|\frac{\varepsilon}{2|A|}+|B|\frac{\varepsilon}{2|B|} = \varepsilon$$

$$\therefore \left|S(P,\phi,\alpha)-A\int_a^b f\,d\alpha - B\int_a^b g\,d\alpha\right| < \varepsilon$$

$$\therefore \int_a^b (Af+Bg)\,d\alpha = A\int_a^b f\,d\alpha + B\int_a^b g\,d\alpha$$

Cor. If $f, g \in R(\alpha)$ on [a, b], then f + g $\in R(\alpha)$ and f – g $\in R(\alpha)$

Proof Taking A = 1 and B = 1 in above theorem we get f + g $\in R(\alpha)$ and A = 1 and B = -1 in above theorem we get f – g $\in R(\alpha)$

Theorem 3.4.2 If $f \in R(\alpha)$ and $f \in R(\beta)$, then $f \in R(A\alpha + B\beta)$ on [a, b], where A and B are real constants then

$$\int_a^b f\,d(A\alpha+B\beta) = A\int_a^b f\,d\alpha + B\int_a^b f\,d\alpha$$

Proof: For a given partition P of [a, b],

$$S(P, f, A\alpha + B\beta) = \sum_{i=1}^{n} f(t_i)(A\Delta\alpha_i + B\Delta\beta_i) = A\sum_{i=1}^{n} f(t_i)\Delta\alpha_i + B\sum_{i=1}^{n} f(t_i)\Delta\beta_i$$
$$= AS(P, f, \alpha) + BS(P, f, \beta)$$

Let $\varepsilon > 0$ be given

since $f \in R(\alpha)$ and $f \in R(\beta)$ there exist partitions P' and P'' such that

$$\left|S(P, f, \alpha) - \int_a^b f\,d\alpha\right| < \frac{\varepsilon}{2|A|} \text{ and } \left|S(P, f, \alpha) - \int_a^b f\,d\beta\right| < \frac{\varepsilon}{2|B|}$$

Where $P' \subseteq P, P'' \subseteq P$

If $P''' = P' \cup P''$ then for any partition P finer than P'''

$$\therefore \left|S(P, f, A\alpha + B\beta) - A\int_a^b f\,d\alpha - B\int_a^b f\,d\beta\right|$$

$$= \left|AS(P, f, \alpha) + BS(P, f, \beta) - A\int_a^b f\,d\alpha - B\int_a^b f\,d\beta\right|$$

$$= \left|AS(P, f, \alpha) - A\int_a^b f\,d\alpha + BS(P, f, \beta) - B\int_a^b f\,d\beta\right|$$

$$\le |A|\left|S(P, f, \alpha) - \int_a^b f\,d\alpha\right| + |B|\left|S(P, f, \beta) - B\int_a^b f\,d\beta\right|$$

$$< |A|\frac{\varepsilon}{2|A|} + |B|\frac{\varepsilon}{2|B|} = \varepsilon$$

$$\therefore \int_a^b f\,d(A\alpha + B\beta) = A\int_a^b f\,d\alpha + B\int_a^b f\,d\alpha$$

Theorem 3.4.3 If $f \in R(\alpha)$ on [a, c] and [c, b], then $f \in R(\alpha)$ on [a, b] and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$

Proof: Let P be a partition of [a, b] such that $c \in P$.

Then $P' = P \cap [a,c]$ and $P'' = P \cap [c,b]$ are partition of {a, c] and [c, b] respectively.

Then Riemann – Stieltjes sums for these partition

$S(P, f, \alpha) = S(P', f, \alpha) + S(P'', f, \alpha)$

Let $\varepsilon > 0$ be given, there is a partition P'_ε of $[a,c]$ and a partition P''_ε of $[c,b]$

$$\left| S(P', f, \alpha) - \int_a^c f d\alpha \right| < \frac{\in}{2} \text{ and } \left| S(P'', f, \alpha) - \int_c^b f d\alpha \right| < \frac{\in}{2}$$

Where $P'_\varepsilon \subset P', P''_\varepsilon \subset P''$

Then $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$ is a partition of [a, b] such that $P_\varepsilon \subset P$

$$\left| S(P, f, \alpha) - \int_a^c f d\alpha - \int_c^b f d\alpha \right| = \left| S(P', f, \alpha) + S(P'', f, \alpha) - \int_a^c f d\alpha - \int_c^b f d\alpha \right|$$

$$= \left| S(P', f, \alpha) - \int_a^c f d\alpha + S(P'', f, \alpha) - \int_c^b f d\alpha \right|$$

$$\le \left| S(P', f, \alpha) - \int_a^c f d\alpha \right| + \left| S(P'', f, \alpha) - \int_c^b f d\alpha \right| < \frac{\in}{2} + \frac{\in}{2} = \varepsilon$$

Hence $\int_a^b f d\alpha$ exists and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$

3.5 Integration By Parts (Partial Integration Theorem)

Theorem 3.5.1 If $f \in R(\alpha)$ on [a, b], then $\alpha \in R(f)$ on [a, b] and

$$\int_a^b f(x) d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x) df(x)$$

Proof: Let $\varepsilon > 0$ be given

Since $f \in R(\alpha)$ on [a, b], there exists a partition P such that

$$\left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < \varepsilon \text{ ....(i)}$$

Let $P = \{a = x_0, x_1, \ldots.., x_n = b\}$ be a partition of [a, b].

Choose $t_1, t_2, \ldots\ldots, t_n$ such that $x_{i-1} \le t_i \le x_i$ and $t_0 = a, t_{n+1} = b$.

So that $t_{i-1} \le x_i \le t_i$. Clearly $Q = \{a = t_0, t_1, t_2, \ldots\ldots, t_{n+1} = b\}$ is also a partition of [a, b].

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i = \sum_{i=1}^{n} f(t_i)(\alpha(x_i) - \alpha(x_{i-1}))$$

$$= f(t_1)(\alpha(x_1) - \alpha(x_0)) + f(t_2)(\alpha(x_2) - \alpha(x_1)) + \ldots\ldots + f(t_n)(\alpha(x_n) - \alpha(x_{n-1}))$$

$$= -\alpha(x_0)[f(t_1) - f(t_0)] - \alpha(x_1)[f(t_2) - f(t_1)] - \ldots\ldots\ldots - \alpha(x_n)[f(t_{n+1}) - f(t_n)]$$
$$- \alpha(x_0)f(t_0) + \alpha(x_n)f(t_{n+1})$$
$$[add \quad and \quad subtract\, \alpha(x_0)f(t_0), \alpha(x_n)f(t_{n+1})]$$

$$= \alpha(b)f(b) - \alpha(a)f(a) - \sum_{i=1}^{n+1} \alpha(x_{i-1})[f(t_i) - f(t_{i-1})]$$

$$\therefore S(P, f, \alpha) = \alpha(b)f(b) - \alpha(a)f(a) - S(Q, \alpha, f)$$

$= A - S(Q, \alpha, f)$.............. (ii)

From (i) and (ii)

$$\left| A - S(Q, \alpha, f) - \int_a^b f d\alpha \right| < \varepsilon$$

$$\Rightarrow \left| S(Q, \alpha, f) - \left[A - \int_a^b f d\alpha \right] \right| < \varepsilon$$

$$\Rightarrow \alpha \in R(f)$$

Also $\int_a^b \alpha(x) df(x)$ = A - $\int_a^b f(x) d\alpha(x)$

From (ii) we have

$\int_a^b \alpha(x) df(x)$ = $f(b)\alpha(b) - f(a)\alpha(a)$ - $\int_a^b f(x) d\alpha(x) \therefore$

$$\int_a^b f(x) d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x) df(x)$$

3.6 Change of Variables

Theorem 3.6.1 If $f \in R(\alpha)$ on [a, b] and g is strictly monotonically increasing function that maps of interval [c, d] onto [a, b]. Let h and β be defined on [c, d] by $h(x) = f(g(x))$ and $\beta(x) = \alpha(g(x))$

Then $h \in R(\beta)$ and $\int_c^d h d\beta = \int_a^b f d\alpha$

Proof: Since g is strictly monotonically increasing and onto function therefore $g:[c,d] \to [a,b]$ is an invertible function.

$\therefore$ Corresponding to every partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of [a, b], there exists a partition $P' = \{c = y_0, y_1, \ldots, y_n = d\}$ of [c, d] such that $g^{-1}(x_i) = y_i$ .In fact, we can write $g^{-1}(P) = P'$

Since $f \in R(\alpha)$, therefore for given $\varepsilon > 0$, there exists a partition P_ε of $[a,b]$ such that $P_\varepsilon \subset P$

$$\left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < \varepsilon \text{ .... (i)}$$

Let $g^{-1}(P_\varepsilon) = P'_\varepsilon$ be the corresponding partition of $[c,d]$ such that $P'_\varepsilon \subset P'$

From a Riemann – Stieltjes sum $S(P',h,\beta)=\sum_{i=1}^{n} h(t_i)\Delta\beta_i$

Where $y_{i-1}\le t_i\le y_i, \Delta\beta_i=\beta(y_i)-\beta(y_{i-1})$

$$\Rightarrow S(P',h,\beta)=\sum_{i=1}^{n} h(t_i)[\beta(y_i)-\beta(y_{i-1})]$$

$$=\sum_{i=1}^{n} f(g(t_i))[\alpha(g(y_i))-\alpha(g(y_{i-1}))]$$

Take $u_i=g(t_i), x_i=g(y_i)$

$$\Rightarrow S(P',h,\beta)=\sum_{i=1}^{n} f(u_i)[\alpha(x_i)-\alpha(x_{i-1})] = S(P,f,\alpha)$$

From (i) $\left|S(P',h,\beta)-\int_a^b f d\alpha\right|<\varepsilon$

$\therefore\ h\in R(\beta)$ and $\int_c^d h d\beta=\int_a^b f d\alpha$

3.7 Reduction to Riemann Integral

Theorem 3.7.1 If $f \in R(\alpha)$ on [a, b] and let α has a continuous derivative α' on [a, b]. Then the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

Proof: Let $g(x) = f(x)\alpha'(x)$

For any partition $P = \{a = x_0, x_1, \ldots.., x_n = b\}$ of [a, b]

Consider Riemann sum

$$S(P,g) = \sum_{i=1}^{n} g(t_i)\Delta x_i = \sum_{i=1}^{n} f(t_i)\alpha'(t_i)\Delta x_i \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Using the same partition P, consider the Riemann – Stieltjes sum

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i = \sum_{i=1}^{n} f(t_i)[\alpha(x_i) - \alpha(x_{i-1})] \ldots\ldots\ldots \text{(ii)}$$

Using mean value theorem we can write

$$\frac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} = \alpha'(c_k),\ x_{i-1} < c_k < x_i$$

From(ii)

$$S(P, f, \alpha)$$

$$= \sum_{i=1}^{n} f(t_i)[x_i - x_{i-1}]\alpha'(c_k) = \sum_{i=1}^{n} f(t_i)\alpha'(c_k)\Delta x_i \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

(iii) - (i) we get

$$S(P, f, \alpha) - S(P, g) = \sum_{i=1}^{n} f(t_i)[\alpha'(c_k) - \alpha'(t_i)]\Delta x_i \ldots\ldots\ldots\ldots\text{(iv)}$$

Since f is bounded on [a, b], therefore M > 0 such that

$\left|f(x)\right| \le M, x \in [a,b]$ since α' is continuous on closed interval [a, b],

therefore it is uniform continuous on [a, b].

$\therefore$ for given $\varepsilon > 0$, there exist $\delta > 0$ such that

$|\alpha'(x) - \alpha'(y)| < \dfrac{\varepsilon}{2M(b-a)}$ whenever $|x - y| < \delta$

If we take a partition P_ε with norm $\|P_\varepsilon\| < \delta$ then for any finer partition P

we have $|\alpha'(c_k) - \alpha'(t_i)| < \dfrac{\varepsilon}{2M(b-a)}$ from (iv)

$$|S(P, f, \alpha) - S(P, g)| = \sum_{i=1}^{n} |f(t_i)||\alpha'(c_k) - \alpha'(t_i)| \Delta x_i < M.\frac{\varepsilon}{2M(b-a)} \sum_{i=1}^{n} \Delta x_i$$

$$= M.\frac{\varepsilon}{2M(b-a)}(b-a)$$

$$\therefore |S(P, f, \alpha) - S(P, g)| < \frac{\varepsilon}{2} \quad \text{............ (v)}$$

Since $f \in R(\alpha)$ on [a, b], there exists a partition P'_ε such that P finer than P'_ε therefore

$$\left|S(P, f, \alpha) - \int_a^b f d\alpha\right| < \frac{\varepsilon}{2} \quad \text{................ (vi)}$$

If $P''_\varepsilon = P_\varepsilon \cup P'_\varepsilon$, then both the inequalities (v) and (vi) hold for every partition P finer than P''_ε $\therefore$

$$\left|S(P, g) - \int_a^b f d\alpha\right| = \left|\left[S(P, f, \alpha) - \int_a^b f d\alpha\right] - [S(P, f, \alpha) - S(P, g)]\right|$$

$$\leq \left| S(P,f,\alpha)-\int_a^b f d\alpha \right| + \left| S(P,f,\alpha)-S(P,g) \right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \therefore$$

$\int_a^b g(x)dx = \int_a^b f d\alpha$ exists and $\int_a^b f(x)\alpha'(x)dx = \int_a^b f d\alpha$

3.8 Step Functions as Integrators

If α is constant in [a, b] then $S(P,f,\alpha)=0$ or $\int_a^b f d\alpha = 0$

However, if α is constant except for a jump discontinuity at one point, then $\int_a^b f d\alpha$ need not exists, if it exist than its value may not be zero.

Definition Step function- A function $\alpha:[a,b]\to R$ is called step function if there is a partition of [a, b]. Let $P=\{a=x_0,x_1,....,x_n=b\}$ such that α is constant in (x_{i-1},x_i)

Jump: Difference between $\alpha\left(x_i^{+}\right)-\alpha\left(x_i^{-}\right)$ **is called jump denoted by** α_i

Theorem 3.8.1 Let a < c < b. Define $\alpha \in [a,b]$ as follows:

$\alpha(x)=\begin{cases}\alpha(a), & a \le x < c \\ \alpha(b), & c < x \le b\end{cases}$, The values $\alpha(a),\ \alpha(b)$ and $\alpha(c)$ are arbitrary.

Let $f \in [a,b]$ be defined in such a way that at least one of the function f or α is continuous from left at c and at least one is continuous from the right at c. Then $f \in R(\alpha)$ on [a, b] and we have $\int_a^b f d\alpha = f(c)\left[\alpha(c^+)-\alpha(c^-)\right]$.

Proof: If $c \in P$, then $S(P, f, \alpha)$ is zero except the two terms which arise from the subinterval separated by c, so that

$$S(P, f,\alpha)= f(x_{i-1})\left[\alpha(c)-\alpha(c^-)\right]+ f(x_i)\left[\alpha(c^+)-\alpha(c)\right],$$

$$x_{i-1} < c < x_i$$

$$\begin{aligned}
&S(P, f,\alpha)- f(c)\left[\alpha(c^+)-\alpha(c^-)\right] \\
&= f(x_{i-1})\left[\alpha(c)-\alpha(c^-)\right]+ f(x_i)\left[\alpha(c^+)-\alpha(c)\right]- f(c)\left[\alpha(c^+)-\alpha(c^-)\right] \\
&= f(x_{i-1})\left[\alpha(c)-\alpha(c^-)\right]+ f(x_i)\left[\alpha(c^+)-\alpha(c)\right] \\
&- f(c)\left[\alpha(c^+)-\alpha(c)+\alpha(c)-\alpha(c^-)\right] \\
&=\left[f(x_{i-1})- f(c)\right]\left[\alpha(c)-\alpha(c^-)\right]+\left[f(x_i)- f(c)\right]\left[\alpha(c^+)-\alpha(c)\right]
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left|S(P, f,\alpha)- f(c)\left[\alpha(c^+)-\alpha(c^-)\right]\right| \\
&=\left|\left[f(x_{i-1})- f(c)\right]\left[\alpha(c)-\alpha(c^-)\right]+\left[f(x_i)- f(c)\right]\left[\alpha(c^+)-\alpha(c)\right]\right| \\
&\le \left|\left[f(x_{i-1})- f(c)\right]\left[\alpha(c)-\alpha(c^-)\right]\right| \\
&+\left|\left[f(x_i)- f(c)\right]\left[\alpha(c^+)-\alpha(c)\right]\right| \qquad \text{.............} \text{ (i)}
\end{aligned}$$

Case1. If f is continuous at c, given $\varepsilon > 0$ there exist $\delta > 0$ such that $\|P\| < \delta$

$\therefore \left|f\left(x_{i-1}\right)-f(c)\right|<\varepsilon$ and $\left|f\left(x_i\right)-f(c)\right|<\varepsilon$

From (i) we have

$$\left|S(P, f, \alpha)-f(c)\left[\alpha\left(c^{+}\right)-\alpha\left(c^{-}\right)\right]\right|<\varepsilon\left|\alpha\left(c^{+}\right)-\alpha\left(c^{-}\right)\right| \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Case 2. If f is discontinuous at c and α is continuous at c.

$\Rightarrow \alpha(c)=\alpha\left(c^{-}\right)$ And $\alpha(c)=\alpha\left(c^{+}\right)$

From (i) $\left|S(P, f, \alpha)-f(c)\left[\alpha\left(c^{+}\right)-\alpha\left(c^{-}\right)\right]\right|<0<\varepsilon$ .... (iii)

Case 3. If f is continuous from the left and discontinuous from the right at c then α will be continuous from right so that $\alpha(c)=\alpha\left(c^{+}\right)$. Then from (i) we have

$$\left|S(P, f, \alpha)-f(c)\left[\alpha\left(c^{+}\right)-\alpha\left(c^{-}\right)\right]\right|<\varepsilon\left|\alpha(c)-\alpha\left(c^{-}\right)\right| \ldots\ldots\ldots \text{(iv)}$$

Case 4. If f is continuous from the right and discontinuous from the left at c then α will be continuous from left so that $\alpha(c)=\alpha\left(c^{-}\right)$. Then from (i) we have

$$\left|S(P, f, \alpha)-f(c)\left[\alpha\left(c^{+}\right)-\alpha\left(c^{-}\right)\right]\right|<\varepsilon\left|\alpha\left(c^{+}\right)-\alpha(c)\right| \ldots\ldots\ldots\ldots \text{(v)}$$

$\therefore$ from (ii) , (ii), (iv), (v) we get

$$\underset{\mu(P)\to 0}{Lt} S(P, f, \alpha)=f(c)\left[\alpha\left(c^{+}\right)-\alpha\left(c^{-}\right)\right]$$

$$\Rightarrow \int_a^b f d\alpha=f(c)\left[\alpha\left(c^{+}\right)-\alpha\left(c^{-}\right)\right]$$

3.9 Reduction Of Riemann – Stieltjes Integral to a finite Sum

Theorem 3.9.1 Let α be a step function defined on [a, b] with $f:[a,b]\to R$ such that not both f and α are discontinuous from right or left at each jump point α_i at x_i. Then $f\in R(\alpha)$ on [a, b] and

$\int_a^b f d\alpha=\sum_{i=1}^{n} f(x_i)\alpha_i$ where $x_1, x_2,\ldots, x_n$ are jump point of α

Proof: Let $P=\{a=x_0, x_1,\ldots, x_n=b\}$ be a partition of [a, b]

$$\int_a^b f d\alpha=\int_a^{x_1} f d\alpha+\int_{x_1}^{x_2} f d\alpha+\ldots\ldots+\int_{x_{n-1}}^{b} f d\alpha$$

$$\int_a^b f d\alpha=f(x_1)\alpha_1+f(x_2)\alpha_2+\ldots\ldots+f(x_n)\alpha_n=\sum_{i=1}^{n} f(x_i)\alpha_i$$

$\therefore\ f\in R(\alpha)$

Theorem 3.9.2. Every finite sum can be written as a Riemann – Stieltjes integral. In fact, given a sum $\sum_{i=1}^{n} a_i$ define f be a function on [0. n] for which $f(i)=a_i$ and continuous from left at $i=1,2,\ldots,n$ with f (0) = 0. Then

$$\sum_{i=1}^{n} a_i=\sum_{i=1}^{n} f(i)=\int_0^n f(x)d[x]$$

Proof: As $\alpha(x)$ =[x] = greatest integer function.

$\therefore$ Jump for [x] is 1 at each jump point i.e. $\alpha(x_i)=1=\alpha_i$

$$\int_0^n f(x)d[x] = \sum_{i=1}^n f(i)\alpha_i = \sum_{i=1}^n f(i) = \sum_{i=1}^n a_i$$

3.10 Euler's Summation Formula

Theorem 3.10.1 If f has a continuous derivative f' on [a, b], then we have

$$\sum_{a<x\le b} f(x) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx + f(a)((a)) - f(b)((b))$$

Where $((x)) = x - [x]$.

When a and b are integers, this becomes

$$\sum_{x=a}^b f(x) = \int_a^b f(x)dx + \int_a^b f'(x)\left(x - [x] - \frac{1}{2}\right)dx + \frac{f(a)+f(b)}{2}$$

Proof: Using integration by parts, we get

$$\int_a^b f(x)d(x-[x]) = \left|f(x)(x-[x])\right|_a^b - \int_a^b (x-[x])d(f(x))$$

$$\Rightarrow \int_a^b f(x)d(x-[x]) + \int_a^b (x-[x])d(f(x)) \quad \text{........ (i)}$$

$$= f(b)(b-[b]) - f(a)(a-[a])$$

As Riemann – Stieltjes is a linear function of the integrator, we have

$$\int_a^b f(x)d(x-[x]) = \int_a^b f(x)dx - \int_a^b f(x)d[x] \quad \text{....... (ii)}$$

Since f' is continuous on [a, b], we have

$$\int_a^b x-[x]d(f(x)) = \int_a^b f'(x)(x-[x])dx \quad \text{..........(iii)}$$

From (i) , (ii) and (iii), we have

$$\int_a^b f(x)dx - \int_a^b f(x)d[x] + \int_a^b f'(x)(x-[x])dx = f(b)\big(b-[b]\big) - f(a)\big(a-[a]\big)$$

$$\int_a^b f(x)d[x] = \int_a^b f(x)dx + \int_a^b f'(x)(x-[x])dx + f(a)\big(a-[a]\big) - f(b)\big(b-[b]\big)$$

As greatest integer function has jumps at integral points has jumps 1

As every finite sum can be written as Riemann – Stieltjes integral by we have $\int_a^b f(x)d[x] = \sum_{a<x\le b} f(x)$

so

$$\sum_{a<x\le b} f(x) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx + f(a)\big(a-[a]\big) - f(b)\big(b-[b]\big)$$

If a and b are integers, then ((a)) = 0 and ((b)) = 0 and

$\sum_{x=a}^{b} f(x) = \sum_{a<x\le b} f(x) + f(a)$ we have

$$\sum_{x=a}^{b} f(x) - f(a) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx$$

$$\Rightarrow \sum_{x=a}^{b} f(x) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx + f(a) - \frac{1}{2}\int_a^b f'(x)dx + \frac{1}{2}\int_a^b f'(x)dx$$

$$\Rightarrow \sum_{x=a}^{b} f(x) = \int_a^b f(x)dx + \int_a^b f'(x)((x))dx + f(a) - \frac{1}{2}\int_a^b f'(x)dx + \frac{1}{2}[f(b) - f(a)]$$

$$\Rightarrow \sum_{x=a}^{b} f(x) = \int_a^b f(x)dx + \int_a^b f'(x)\left(x-[x]-\frac{1}{2}\right)dx + \frac{f(a)+f(b)}{2}$$

3.11 Comparison Theorem

Theorem 3.11.1 If $f \in R(\alpha)$ on [a, b], then

$$m[\alpha(b)-\alpha(a)] \le \int_a^b f d\alpha \le M[\alpha(b)-\alpha(a)]$$ where m and M are bounds of 'f' in [a, b]

Proof: Since $f \in R(\alpha)$ on [a, b], then

$$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha$$

Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be any partition of [a, b] and m_i, M_i be bound of 'f' in $[x_{i-1}, x_i]$

Then $m \le m_i \le M_i \le M$

$$\Rightarrow \sum_{i=1}^{n} m \Delta\alpha_i \le \sum_{i=1}^{n} m_i \Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i \le \sum_{i=1}^{n} M \Delta\alpha_i$$

$$\Rightarrow m \sum_{i=1}^{n} \Delta\alpha_i \le L(P, f, \alpha) \le U(P, f, \alpha) \le M \sum_{i=1}^{n} \Delta\alpha_i$$

$$\Rightarrow m[\alpha(b)-\alpha(a)] \le L(P, f, \alpha) \le U(P, f, \alpha) \le M[\alpha(b)-\alpha(a)] \quad \text{.......(i)}$$

We know that $L(P, f, \alpha) \le \underline{\int_a^b} f d\alpha \le \overline{\int_a^b} f d\alpha \le U(P, f, \alpha)$

$L(P, f, \alpha) \leq \int_a^b f d\alpha \leq U(P, f, \alpha)$............(ii)

From (i) and (ii) we have $m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$

Theorem 3.11.2 If $f(x) \geq 0 \ \forall \ x \in [a, b]$, then $\int_a^b f d\alpha \geq 0$

Proof: We know that $m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$.

Since $f(x) \geq 0 \Rightarrow m \geq 0$

$\Rightarrow m[\alpha(b) - \alpha(a)] \geq 0$

$\Rightarrow \int_a^b f d\alpha \geq 0$

Theorem 3.11.3 If $f(x) \leq 0 \ \forall \ x \in [a, b]$, then $\int_a^b f d\alpha \leq 0$

Proof: We know that $m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$.

Since $f(x) \leq 0 \Rightarrow M \leq 0$

$\Rightarrow M[\alpha(b) - \alpha(a)] \leq 0$

$\Rightarrow \int_a^b f d\alpha \leq 0$

Cor.3.11.1 If $f_1(x) \leq f_2(x), \ \forall x \in [a, b]$ then $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$

If $f_1(x) \leq f_2(x) \Rightarrow f_1(x) - f_2(x) \leq 0 \Rightarrow (f_1 - f_2)x \leq 0$

$\therefore \int_a^b (f_1 - f_2) d\alpha \leq 0$

$$\Rightarrow \int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha$$

Cor.3.11.2 If $f_1(x) \ge f_2(x)$, $\forall x \in [a,b]$ then $\int_a^b f_1 d\alpha \ge \int_a^b f_2 d\alpha$

3.12 First mean value theorem for Riemann – Stiltjes Integrals

Theorem 3.12.1 If $f \in R(\alpha)$ on [a, b] and α is monotonically increasing on [a, b], then $\int_a^b f d\alpha = \mu[\alpha(b)-\alpha(a)]$ for some μ, where $m \le \mu \le M$.

Further if f is continuous on [a, b], then there exist $c \in [a,b]$ such that

$$\int_a^b f d\alpha = f(c)[\alpha(b)-\alpha(a)]$$

Proof: we known that

$$m[\alpha(b)-\alpha(a)] \le \int_a^b f d\alpha \le M[\alpha(b)-\alpha(a)] \text{.............(i)}$$

Since $m \le \mu \le M$

$$\Rightarrow m[\alpha(b)-\alpha(a)] \le \mu[\alpha(b)-\alpha(a)] \le M[\alpha(b)-\alpha(a)] \qquad \text{................(ii)}$$

From (i) and (ii) we have $\int_a^b f d\alpha = \mu[\alpha(b)-\alpha(a)]$

Now as f is continuous on [a, b], therefore by intermediate value theorem, there exist $c \in [a,b]$ such that $\mu = f(c)$. Put in (ii) we have

$$\int_a^b f d\alpha = f(c)[\alpha(b)-\alpha(a)]$$

3.13 Second mean value theorem for Riemann – Stiltjes Integrals

Theorem 3.13.1 If f is monotonically increasing on [a, b] and α is continuous on [a, b], then there exist $c \in [a,b]$ such that

$$\int_a^b f d\alpha = f(a)\int_a^c d\alpha + f(b)\int_c^b d\alpha$$

Proof. By integration by parts, we have

$$\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df \qquad \text{..........(i)}$$

Also we have $\int_a^b \alpha df = \alpha(c)[f(b) - f(a)]$

Put in (i) we have

$$\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \alpha(c)[f(b) - f(a)]$$

$$= f(a)[\alpha(c) - \alpha(a)] + f(b)[\alpha(b) - \alpha(c)]$$

$$= f(a)\int_a^c d\alpha + f(b)\int_c^b d\alpha$$

3.14 Fundamental Theorem of Integral Calculus

Theorem 3.14.1 Let α be a function of bounded variation on [a, b] and $f \in R(\alpha)$ on [a, b].

Let $F(x) = \int_a^x f d\alpha$, $x \in [a,b]$

Then (i) F is of bounded variation on [a, b]

(ii) Every point of continuity of α is also a point of continuity of F

(iii) if α is monotonically increasing on [a, b], then

$F'(x)$ Exist at each point x in [a, b] where $\alpha'(x)$ exists and f is continuous. Also for such x, we have

$F'(x) = f(c)\alpha'(x), c \in [a,b]$

Proof: Let $x, y \in [a,b]$ and $x \neq y$

Then $F(y) - F(x) = \int_a^y f d\alpha - \int_a^x f d\alpha$

$= \int_a^y f d\alpha + \int_x^a f d\alpha = \int_x^y f d\alpha$ ....(i)

By first mean value theorem for R-S integral, we have

$\int_x^y f d\alpha = \mu[\alpha(y) - \alpha(x)]$ where $m \leq \mu \leq M$ ....(ii)

From (i) and (ii) we have $F(y) - F(x) = \mu[\alpha(y) - \alpha(x)]$...........(iii)

(i) Since α be a function of bounded variation on [a, b] , then from (iii) we get F is also bounded variation on [a, b].

(ii) Let α be continuous at c, where $c \in [a,b]$, therefore for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\alpha(x) - \alpha(c)| < \dfrac{\varepsilon}{|\mu|}$ whenever $|x - c| < \delta$ From (iii) we have

$$|F(x) - F(c)| = |\mu||\alpha(x) - \alpha(c)| < |\mu| \frac{\varepsilon}{|\mu|} = \varepsilon$$

Hence F is continuous at c

(iii) $F'(x) = \lim\limits_{y \to x} \dfrac{F(y) - F(x)}{y - x} = \mu \lim\limits_{y \to x} \dfrac{\alpha(y) - \alpha(x)}{y - x} = \mu\alpha'(x)$

$= f(c)\alpha'(x)$

3.15 Integrators Of Bounded Variation

Theorem 3.15.1 Assume that α is of bounded variation on $[a,b]$. Let $V(x)$ denote the total variation of α on $[a,x]$ if $a < x \le b$, and $V(a) = 0$. Let f be defined and bounded on $[a,b]$.

If $f \in R(\alpha)$ on $[a,b]$, then $f \in R(V)$ on $[a,b]$.

Proof: If $V(b) = 0$, then V is constant and the result is trivial. Suppose therefore, that $V(b) > 0$.

Since f is bounded on $[a,b]$. Let $|f(x)| \le M \ \forall x \in [a,b]$

Since $f \in R(\alpha)$ on $[a,b]$, given $\varepsilon > 0$, choose a partition P' so that

For any finer P and $t_i, t'_i \in [x_{i-1}, x_i]$

We have $\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \frac{\varepsilon}{8}$ and $\left|\sum_{i=1}^{n} f(t'_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \frac{\varepsilon}{8}$

$$\Rightarrow \left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \sum_{i=1}^{n} f(t'_i)\Delta\alpha_i\right|$$

$$= \left|\left[\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right] - \left[\sum_{i=1}^{n} f(t'_i)\Delta\alpha_i - \int_a^b f d\alpha\right]\right|$$

$$\leq \left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| + \left|\sum_{i=1}^{n} f(t'_i)\Delta\alpha_i - \int_a^b f d\alpha\right|$$

$$< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4} \quad \text{.................(i)}$$

Since $V(b)$ is total variation of α .

$$\therefore V(b) = \sup_P \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1})\right| = \sup_P \sum_{i=1}^{n} |\Delta\alpha_i|$$

$$\therefore \sum_{i=1}^{n} |\Delta\alpha_i| > V(b) - \frac{\varepsilon}{4M}$$

$$\Rightarrow V(b) < \sum_{i=1}^{n} |\Delta\alpha_i| + \frac{\varepsilon}{4M} \quad \text{...................(ii)}$$

To prove $U(P, f, V) - L(P, f, V) < \varepsilon$

$\therefore$ We shall prove two inequalities

$\sum_{i=1}^{n} |M_i - m_i| \left(\Delta V_i - |\Delta \alpha_i|\right) < \frac{\varepsilon}{2}$ and $\sum_{i=1}^{n} |M_i - m_i| |\Delta \alpha_i| < \frac{\varepsilon}{2}$

We note that $\Delta V_i - |\Delta \alpha_i| \geq 0$

$$\sum_{i=1}^{n} |M_i - m_i| \left(\Delta V_i - |\Delta \alpha_i|\right) = \sum_{i=1}^{n} | f(x_i) - f(x_{i-1}) | \left(\Delta V_i - |\Delta \alpha_i|\right)$$

$$\leq 2M \sum_{i=1}^{n} \left(\Delta V_i - |\Delta \alpha_i|\right)$$

$$= 2M \sum_{i=1}^{n} \left[V(b) - V(a) - |\Delta \alpha_i|\right]$$

$$= 2M \sum_{i=1}^{n} \left[V(b) - |\Delta \alpha_i|\right]$$

$< 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}$ {from (ii)} ....(iii)

By definition of positive and negative variations

$$A(P) = \{i : \Delta \alpha_i > 0\},\ B(P) = \{i : \Delta \alpha_i < 0\}$$

Choose $h = \frac{\varepsilon}{4V(b)}$

$$M_i - m_i = \sup\{ f(x) - f(y) : x, y \in [a, b]\}$$

If $i \in A(P)$ then $t_i > t'_i \Rightarrow f(t_i) > f(t'_i)$

$$\therefore M_i - m_i - h < f(t_i) - f(t'_i)$$

$\therefore\ M_i - m_i < f(t_i) - f(t'_i) + h$ ....(iv)

If $i \in B(P)$ then $t_i > t'_i \Rightarrow f(t_i) < f(t'_i)$

$\therefore\ M_i - m_i - h < f(t'_i) - f(t_i)$

$\therefore\ M_i - m_i < f(t'_i) - f(t_i) + h$ ....(v)

$$\sum_{i=1}^{n}[M_i - m_i]|\Delta\alpha_i| = \sum_{i \in A(P)}[M_i - m_i]|\Delta\alpha_i| + \sum_{i \in B(P)}[M_i - m_i]|\Delta\alpha_i|$$

$$< \sum_{i \in A(P)}\left[f(t_i) - f(t'_i) + h\right]|\Delta\alpha_i| + \sum_{i \in B(P)}\left[f(t'_i) - f(t_i) + h\right]|\Delta\alpha_i|$$

{from (iv) and (v)}

$$= \sum_{i \in A(P)}\left[f(t_i) - f(t'_i)\right]|\Delta\alpha_i| + \sum_{i \in B(P)}\left[f(t'_i) - f(t_i)\right]|\Delta\alpha_i| + h\sum_{i=1}^{n}|\Delta\alpha_i|$$

$$= \sum_{i \in A(P)}\left[f(t_i) - f(t'_i)\right]\Delta\alpha_i + \sum_{i \in B(P)}\left[f(t_i) - f(t'_i)\right]\Delta\alpha_i + h\sum_{i=1}^{n}|\Delta\alpha_i|$$

$$= \sum_{i=1}^{n}\left[f(t_i) - f(t'_i)\right]\Delta\alpha_i + h\sum_{i=1}^{n}|\Delta\alpha_i| < \frac{\varepsilon}{4} + h\sum_{i=1}^{n}|\Delta V_i|$$

$\{\because\ \Delta V_i - |\Delta\alpha_i| \geq 0 \Rightarrow |\Delta\alpha_i| \leq \Delta V_i\}$

$$= \frac{\varepsilon}{4} + h[V(b) - V(a)] = \frac{\varepsilon}{4} + h.V(b)$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4V(b)}.V(b) = \frac{\varepsilon}{2}$$

$$\therefore U(P, f, V) - L(P, f, V) = \sum_{i=1}^{n} [M_i - m_i] \Delta V_i$$

$$= \sum_{i=1}^{n} [M_i - m_i] [\Delta V_i - |\Delta \alpha_i| + |\Delta \alpha_i|]$$

$$= \sum_{i=1}^{n} [M_i - m_i] [\Delta V_i - |\Delta \alpha_i|] + \sum_{i=1}^{n} [M_i - m_i] |\Delta \alpha_i| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

3.16 Some Theorems

Theorem 3.16.1 Assume that α is monotonic increasing on [a, b]. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on [a, b] and if f(x) $\leq$ g(x) for all x in [a, b], then we have $\int_a^b f d\alpha \leq \int_a^b g d\alpha$.

Proof. For every partition P of [a, b] then

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i) \Delta x_i \leq \sum_{i=1}^{n} g(t_i) \Delta x_i = S(P, g, \alpha) \quad \text{.....(i)}$$

Since $f \in R(\alpha)$ then for a partition P'

, $\left| S(P', f, \alpha) - \int_a^b f d\alpha \right| < \frac{\varepsilon}{2}$

$$\Rightarrow \int_a^b f d\alpha - \frac{\varepsilon}{2} < S(P', f, \alpha) < \int_a^b f d\alpha + \frac{\varepsilon}{2} \quad \text{..............(ii)}$$

Since $f \in R(\alpha)$ then for a partition P'',

$$\left| S(P'', g, \alpha) - \int_a^b g d\alpha \right| < \frac{\varepsilon}{2} \quad \text{.......(iii)}$$

$$\Rightarrow \int_a^b g d\alpha - \frac{\varepsilon}{2} < S(P'', g, \alpha) < \int_a^b g d\alpha + \frac{\varepsilon}{2}$$

Let $P = P' \cup P''$ so we have

$$\int_a^b f d\alpha - \frac{\varepsilon}{2} < S(P, f, \alpha) < \int_a^b f d\alpha + \frac{\varepsilon}{2} \text{ and}$$

$$\int_a^b g d\alpha - \frac{\varepsilon}{2} < S(P, g, \alpha) < \int_a^b g d\alpha + \frac{\varepsilon}{2}$$

Subtract these we get

$$\int_a^b f d\alpha - \frac{\varepsilon}{2} - \int_a^b g d\alpha + \frac{\varepsilon}{2} < S(P, f, \alpha) - S(P, g, \alpha) \le 0 \text{ from (i)}$$

$$\therefore \int_a^b f d\alpha - \int_a^b g d\alpha \le 0$$

$$\Rightarrow \int_a^b f d\alpha \le \int_a^b g d\alpha$$

Theorem 3.16.2 Let $f \in R(\alpha)$ on [a, b] and $m \le f(x) \le M \ \forall x \in [a, b]$. Let α is monotonic increasing on [a, b]. Let $\phi : [m, M] \to R$ be a continuous function. Then $h = \phi \circ f \in R(\alpha)$

Proof. Let $\varepsilon > 0$ be given. Since ϕ is continuous on [m, M].

$\therefore$ ϕ is uniform continuous on [m, M].

$\therefore$ for given $\varepsilon > 0$, there exist $\delta > 0$ such that

$|\phi(s) - \phi(t)| < \varepsilon$ whenever $|s - t| < \delta$, $s, t \in [m, M]$ ....(i)

We may assume that $\delta < \varepsilon$

Since $f \in R(\alpha)$ on [a, b], therefore given $\delta^2 > 0$, there exists a partition $P = \{a = x_0, x_1, ..., x_n = b\}$ of [a, b] such that

$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2$ ....(ii)

Let $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$, $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$,

$M_i^* = \sup\{h(x) : x \in [x_{i-1}, x_i]\}$ and $m_i^* = \inf\{h(x) : x \in [x_{i-1}, x_i]\}$

We divide the numbers $i = 1, 2, ..., n$ into two classes A and B where

$A = \{i : M_i - m_i < \delta\}, B = \{i : M_i - m_i \geq \delta\}$

When $i \in A$ and $s, t \in [x_{i-1}, x_i]$

$\therefore$ $|f(s) - f(t)| \leq M_i - m_i < \delta$ From (i) $|\phi(f(s)) - \phi(f(t))| < \varepsilon$

$\Rightarrow |(\phi \circ f)(s) - (\phi \circ f)(t)| < \varepsilon$

$\Rightarrow |h(s) - h(t)| < \varepsilon$

$\Rightarrow M_i^* - m_i^* < \varepsilon$ ....(iii)

$$\sum_{i \in A} \left(M_i^* - m_i^*\right) \Delta\alpha_i < \varepsilon \sum_{i \in A} \Delta\alpha_i$$

$$< \varepsilon \sum_{i=1}^{n} \Delta\alpha_i = \varepsilon[\alpha(b)-\alpha(a)]$$..............(iv)

Again $i \in B \Rightarrow M_i - m_i \ge \delta$ i.e. $\delta \le M_i - m_i$

$$\Rightarrow \delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B}(M_i - m_i)\Delta\alpha_i \le \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = U(P, f, \alpha) - L(P, f, \alpha)$$

$$\Rightarrow \sum_{i \in B} \Delta\alpha_i < \delta < \varepsilon$$ ....(v)

Let $K = \sup\{\phi(s) : m \le s \le M\}$ we have

$$|\phi(s) - \phi(t)| \le |\phi(s)| + |\phi(t)| \le k + k = 2k$$

$$\therefore M_i^* - m_i^* \le 2k$$

$$\sum_{i \in B}(M_i - m_i)\Delta\alpha_i \le 2k \sum_{i \in B} \Delta\alpha_i \le 2k \sum_{i=1}^{n} \Delta\alpha_i < 2k\varepsilon$$ {from (v)}

$$U(P, h, \alpha) - L(P, h, \alpha) = \sum_{i \in A}\left(M_i^* - m_i^*\right)\Delta\alpha_i + \sum_{i \in B}\left(M_i^* - m_i^*\right)\Delta\alpha_i$$

$$< \varepsilon[\alpha(b) - \alpha(a)] + 2k\varepsilon$$

$= \varepsilon[\alpha(b) - \alpha(a) + 2k]$ since ε is arbitrary

$$\therefore h = \phi \circ f \in R(\alpha)$$

Theorem 3.16.3 If $f \in R(\alpha)$ on [a, b], then $|f| \in R(\alpha)$ and

$$\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$$

Proof. Let $\phi(t) = |t|$

$$\therefore (\phi \circ f)(t) = \phi(f(t)) = |f(t)| = |f|(t)$$

$\Rightarrow \phi \circ f = |f|$

Since $\phi(t)$ is continuous on [m, M] and $f \in R(\alpha)$ on [a, b]

$\therefore \phi \circ f \in R(\alpha)$ on [a, b]

$\therefore |f| \in R(\alpha)$

Now $x \in [a,b],\ f(x) \le |f|(x)$

$\int_a^b f d\alpha \le \int_a^b |f| d\alpha$ ....(i)

Since $-f(x) \le |f|(x) - \int_a^b f d\alpha \le \int_a^b |f| d\alpha$ ....(ii)

From (i) and (ii) we get $\left|\int_a^b f d\alpha\right| \le \int_a^b |f| d\alpha$

3.17Examples

Example 1 Prove that $\int_a^b d\alpha = \alpha(b) - \alpha(a)$

Sol. Let f = 1, Let $P = \{a = x_0, x_1, x_2, \ldots\ldots, x_n = b\}$ be a partition of [a, b].

We know that $S(P, f, \alpha) = \int_a^b f d\alpha = \sum_{i=1}^{n} f(t_i) \Delta\alpha_i$

$$\therefore\ S(P,1,\alpha)=\int_a^b 1d\alpha = \sum_{i=1}^{n}\Delta\alpha_i = \sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]$$

$$=[\alpha(x_1)-\alpha(x_0)]+[\alpha(x_2)-\alpha(x_1)]+\dots\dots\dots+[\alpha(x_n)-\alpha(x_{n-1})]$$

$$=\alpha(b)-\alpha(a)$$

Example 2 If $f \in R(\alpha)$ on [a, b] and if $\int_a^b f d\alpha = 0$ for every f which is monotonic on [a, b], prove that α must be constant on [a, b].

Sol. By **Theorem 3.8.1**

Example 3 Let $\{a_n\}$ be a sequence of real numbers. For $x \geq 0$, define

$$A(x)=\sum_{n\leq x} a_n = \sum_{n=1}^{[x]} a_n$$

where [x] is the greatest integer function and empty sums interpreted as zero. Let f have continuous derivative in the interval $1 \leq x \leq a$ then

$$\sum_{n\leq x} a_n f(x) = -\int_1^a A(x) f'(x)dx + A(a) f(a)$$

Sol. As f has continuous derivative on [1, a], then

$$\int_1^a A(x) f'(x)dx = |A(x) f(x)|_1^a - \int_1^a f(x)dA(x)$$

$$= A(a) f(a) - A(1) f(1) - \int_1^a f(x)dA(x)$$

Also A(1) = a_1

$$\therefore\ \int_1^a A(x) f'(x)dx = A(a) f(a) - a_1 f(1) - \int_1^a f(x)dA(x)$$

$$\therefore \int_1^a A(x) f'(x)dx = A(a) f(a) - \left(a_1 f(1) + \int_1^a f(x)dA(x) \right) \text{.... (i)}$$

Also $a_1 f(1) + \int_1^a f(x)dA(x) = \sum_{n=1}^{[a]} a_n f(n) = \sum_{n \le x} a_n f(x)$

Put in (i), we have $\int_1^a A(x) f'(x)dx = A(a) f(a) - \sum_{n \le x} a_n f(x)$

$$\Rightarrow \sum_{n \le x} a_n f(x) = -\int_1^a A(x) f'(x)dx + A(a) f(a)$$

Example 4 Using integration by parts prove that the following

(i) $\sum_{\alpha=1}^{n} \frac{1}{\alpha^s} = \frac{1}{n^{s-1}} + s\int_1^n \frac{[x]}{x^{s+1}}dx$ if $s \ne 1$

(ii) $\sum_{k=1}^{n} \frac{1}{k} = \log n - \int_1^n \frac{x-[x]}{x^2}dx + 1$

Sol.

$$\sum_{i=1}^{n} f(i) = \int_0^n f(x)d[x] \Rightarrow \sum_{i=1}^{n} f(i) = f(1) + \sum_{i=2}^{n} f(i) = f(1) + \int_1^n f(x)d[x]$$

(i) $\sum_{\alpha=1}^{n} \frac{1}{\alpha^s} = 1 + \int_1^n x^{-s} d[x] = 1 - \int_1^n [x] d\, x^{-s} + n^{-s}[n] - 1^{-s}[1]$

$$= -\int_1^n [x]\left(\frac{-s}{x^{s+1}} \right)dx + n^{-s} n$$

$$= s\int_1^n [x]\left(\frac{1}{x^{s+1}}\right)dx + n^{1-s}$$

$$\therefore \sum_{\alpha=1}^{n} \frac{1}{\alpha^s} = \frac{1}{n^{s-1}} + s\int_1^n \frac{[x]}{x^{s+1}}dx$$

(ii) $\sum_{k=1}^{n} \frac{1}{k} = 1 + \int_1^n \frac{1}{x} d[x] = 1 - \int_1^n [x] d\, x^{-1} + n^{-1}[n] - 1^{-1}[1]$

$$= -\int_1^n [x]\left(\frac{-1}{x^2}\right)dx + \frac{1}{n} n = \int_1^n [x]\left(\frac{1}{x^2}\right)dx + 1$$

Add and subtract $\int_1^n \frac{1}{x} dx$, we get

$$\sum_{k=1}^{n} \frac{1}{k} = \int_1^n [x]\left(\frac{1}{x^2}\right)dx + 1 - \int_1^n \frac{1}{x} dx + \int_1^n \frac{1}{x} dx = \log n - \log 1 - \int_1^n \left(\frac{1}{x} - \frac{[x]}{x^2}\right)dx$$

+1

$$\therefore \sum_{k=1}^{n} \frac{1}{k} = \log n - \int_1^n \frac{x-[x]}{x^2}dx + 1$$

Example 5 If f' is continuous on [1, 2n], show that

$$\sum_{k=1}^{2n} (-1)^k f(k) = \int_1^{2n} f'(x)\left([x] - 2\left[\frac{x}{2}\right]\right)dx$$

Sol.

$$\sum_{k=1}^{2n} (-1)^k f(k) = -f(1) + f(2) - f(3) + f(4) - \dots\dots\dots - f(2n-1) + f(2n)$$

$$= -[f(1) + f(3) + \dots\dots\dots + f(2n-1)] + [f(2) + f(4) + \dots\dots\dots + f(2n)]$$

$$= -\left[f(1)+f(2)+f(3)+f(4)+\ldots\ldots\ldots+f(2n)\right]+2\left[f(2)+f(4)+\ldots\ldots\ldots+f(2n$$

$$= -\sum_{k=1}^{2n} f(k)+\sum_{k=1}^{n} f(2k) = -f(1)-\sum_{k=2}^{2n} f(k)+\sum_{t=2}^{2n} f(t)$$ {take 2 k = t}

$$= -f(1)-\int_1^{2n} f(x)d[x]+2\int_1^{2n} f(x)d\left[\frac{x}{2}\right]$$

$$= -f(1)-\left\{\left[f(x)[x]\right]_1^{2n}-\int_1^{2n}[x]f'(x)dx\right\}+2\left\{\left[f(x)\left[\frac{x}{2}\right]\right]_1^{2n}-\int_1^{2n}\left[\frac{x}{2}\right]f'(x)dx\right\}$$

$$= -f(1)-\left\{f(2n)[2n]-f(1)-\int_1^{2n}[x]f'(x)dx\right\}+2\left\{f(2n)[n]-0-\int_1^{2n}\left[\frac{x}{2}\right]f'(x)dx\right.$$

$$= \int_1^{2n}[x]f'(x)dx-2\int_1^{2n}\left[\frac{x}{2}\right]f'(x)dx-2nf(2n)+2nf(2n) \therefore$$

$$\sum_{k=1}^{2n}(-1)^k f(k) = \int_1^{2n} f'(x)\left([x]-2\left[\frac{x}{2}\right]\right)dx$$

Example 6 Let $g(x)=\begin{cases} x-[x]-\dfrac{1}{2}, & x\notin z \\ 0, & x\in z\end{cases}$. Also let $h(x)=\int_0^x g(t)dt$. If f' is continuous on [a, b], prove that by Euler's summation formula implies that $\sum_{k=1}^{n} f(k)=\int_1^n f(x)dx-\int_1^n h(x)f'(x)dx+\dfrac{f(1)+f(n)}{2}$

Sol. by Euler's summation formula

$$\sum_{k=1}^{n} f(k)=\int_1^n f(x)dx+\int_1^n g(x)f'(x)dx+\frac{f(1)+f(n)}{2} \Rightarrow$$

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx + \int_1^n f'(x)d\left(h(x)\right) + \frac{f(1)+f(n)}{2}$$

$$\{\because d\left(h(x)\right) = g(x) - g(0) = g(x) - 0\}$$

$$= \int_1^n f(x)dx - \int_1^n h(x)d\left(f'(x)\right) + f'(n)h(n) - f'(1)h(1) + \frac{f(1)+f(n)}{2}$$

$$= \int_1^n f(x)dx - \int_1^n h(x)d\left(f'(x)\right) + \frac{f(1)+f(n)}{2}$$

Example 7 Let f(x) = x and $\alpha(x) = x + [x]$. Find $\int_0^{10} f(x)d\alpha(x)$ by definition.

Sol. Let partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots., , \frac{10n}{n}\right\}$

Then

$$S(P, f, \alpha) = \sum_{i=1}^{10n} f(t_i)\Delta\alpha_i = \sum_{i=1}^{10n} f(t_i)\left[\alpha(x_i) - \alpha(x_{i-1})\right] = \sum_{i=1}^{10n} (t_i)\left[\alpha\left(\frac{i}{n}\right) - \alpha\left(\frac{i-1}{n}\right)\right]$$

$$= \sum_{i=1}^{10n} (t_i)\left[\left(\frac{i}{n}\right) + \left[\frac{i}{n}\right] - \left(\frac{i-1}{n}\right) - \left[\frac{i-1}{n}\right]\right]$$

$$= \sum_{i=1}^{10n} (t_i)\left[\left(\frac{1}{n}\right) + \left[\frac{i}{n}\right] - \left[\frac{i-1}{n}\right]\right]$$

Let $t_i \in [x_{i-1}, x_i]$ i.e. $t_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$ take $t_i = \frac{i}{n}$

$$= \sum_{i=1}^{10n} \frac{i}{n}\left[\left(\frac{1}{n}\right) + \left[\frac{i}{n}\right] - \left[\frac{i-1}{n}\right]\right] = \sum_{i=1}^{10n} \frac{i}{n}\left(\frac{1}{n}\right) + \sum_{i=1}^{10n} \frac{i}{n}\left(\left[\frac{i}{n}\right] - \left[\frac{i-1}{n}\right]\right)$$

$$\int_0^{10} f(x)\,d\alpha(x) = \underset{n\to\infty}{Lt}\; S(P, f, \alpha)$$

$$\underset{n\to\infty}{Lt}\sum_{i=1}^{10n}\frac{i}{n}\left(\frac{1}{n}\right) = \underset{n\to\infty}{Lt}\frac{1}{n^2}\sum_{i=1}^{10n} i = \underset{n\to\infty}{Lt}\frac{1}{n^2}\{1+2+.....+10n\} = \underset{n\to\infty}{Lt}\frac{10n(}{2}$$

$$= \underset{n\to\infty}{Lt}\frac{10\left(10+\frac{1}{n}\right)}{2} = 50$$

$$\underset{n\to\infty}{Lt}\sum_{i=1}^{10n}\frac{i}{n}\left(\left[\frac{i}{n}\right]-\left[\frac{i-1}{n}\right]\right) = 0 + \underset{n\to\infty}{Lt}\sum_{i=n}^{10n}\frac{i}{n}\left(\left[\frac{i}{n}\right]-\left[\frac{i-1}{n}\right]\right)$$

$$= \underset{n\to\infty}{Lt}\frac{n}{n}\left\{\left[\frac{n}{n}\right]-\left[\frac{n-1}{n}\right]\right\} + \frac{n+1}{n}\left\{\left[\frac{n+1}{n}\right]-\left[\frac{n}{n}\right]\right\} + ... + \frac{2n-1}{n}\left\{\left[\frac{2n-1}{n}\right]-\right.$$

$$+\frac{2n}{n}\left\{\left[\frac{2n}{n}\right]-\left[\frac{2n-1}{n}\right]\right\} + \frac{2n+1}{n}\left\{\left[\frac{2n+1}{n}\right]-\left[\frac{2n}{n}\right]\right\} + ... + \frac{3n-1}{n}\left\{\left[\frac{3n-1}{n}\right]-\right.$$

$$+\frac{3n}{n}\left\{\left[\frac{3n}{n}\right]-\left[\frac{3n-1}{n}\right]\right\} + \frac{3n+1}{n}\left\{\left[\frac{3n+1}{n}\right]-\left[\frac{3n}{n}\right]\right\} + ... + \frac{4n-1}{n}\left\{\left[\frac{4n-1}{n}\right]-\right.$$

+............

$$+\frac{9n}{n}\left\{\left[\frac{9n}{n}\right]-\left[\frac{9n-1}{n}\right]\right\} + \frac{9n+1}{n}\left\{\left[\frac{9n+1}{n}\right]-\left[\frac{9n}{n}\right]\right\} + ... + \frac{10n-1}{n}\left\{\left[\frac{10n-1}{n}\right]-\right.$$

$$+\frac{10n}{n}\left\{\left[\frac{10n}{n}\right]-\left[\frac{10n-1}{n}\right]\right\} = 1+2+........+10 = \frac{10(10+1)}{2} = 55$$

$$\therefore \int_0^{10} f(x)\,d\alpha(x) = \underset{n\to\infty}{Lt}\; S(P, f, \alpha) = 50 + 55 = 105$$

Example 8 Let f(x) = x and $\alpha(x) = x^2$. Find $\int_0^1 f\,d\alpha$ by definition.

Sol. Let partition $P=\left\{0,\frac{1}{n},\frac{2}{n},....,\frac{n}{n}\right\}$ and let $t_i\in[x_{i-1},x_i]$ i.e.

$t_i\in\left[\frac{i-1}{n},\frac{i}{n}\right]$ Take $t_i=\frac{i}{n}$. Then

$$\int_0^1 f d\alpha=\underset{n\to\infty}{Lt}\ S(P,f,\alpha)=\underset{n\to\infty}{Lt}\sum_{i=1}^{n} f(t_i)[\alpha(x_i)-\alpha(x_{i-1})]$$

$$=\underset{n\to\infty}{Lt}\sum_{i=1}^{n}\frac{i}{n}\left[\frac{i^2}{n^2}-\frac{(i-1)^2}{n^2}\right]$$

$$=\underset{n\to\infty}{Lt}\sum_{i=1}^{n}\frac{i}{n^3}\left[i^2-i^2-1+2i\right]=\underset{n\to\infty}{Lt}\sum_{i=1}^{n}\frac{i}{n^3}[2i-1]$$

$$=\underset{n\to\infty}{Lt}\frac{2}{n^3}\sum_{i=1}^{n}i^2-\underset{n\to\infty}{Lt}\frac{1}{n^3}\sum_{i=1}^{n}i$$

$$=\underset{n\to\infty}{Lt}\frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)-\underset{n\to\infty}{Lt}\frac{1}{n^3}\left(\frac{n(n+1)}{2}\right)=\frac{2}{3}$$

Example 9 Let f(x) = x and $\alpha(x)=x+[x]$. Find $\int_0^{10} f(x)d\alpha(x)$ by integration by parts.

Sol. $\int_0^{10} f d\alpha=\int_0^{10} x d(x+[x])=\{x(x+[x])\}_0^{10}-\int_0^{10}(x+[x])dx$

$$=200-0-\int_0^{10} x dx-\int_0^{10}[x]dx=200-50-45=105$$

Example 10 Evaluate $\int_0^4 \left(\left[\sqrt{x}\right] + x^2\right) d\sqrt{x}$.

Sol. Let $\sqrt{x} = y$ $\therefore$ $\int_0^2 \left([y] + y^4\right) dy = \int_0^2 [y] dy + \int_0^2 y^4 dy = \frac{37}{5}$

Example 11 Find $\int_{-1}^{1} f d\alpha$, where f(x) = 1, $x \in [-1,1]$ and α is a step function such that continuous $\alpha(x) = \begin{cases} -1, & x \le 0 \\ -1, & x > 0 \end{cases}$.

Sol. $\int_{-1}^{1} f d\alpha = f(0)\left[\alpha\left(0^+\right) - \alpha\left(0^-\right)\right] = 1[-1+1] = 0$

Example 12 If $f(x) = \begin{cases} 3, & x \le 0 \\ 3-4x, & 0 < x < 1 \\ -1, & x \ge 1 \end{cases}$ and $\alpha(x) = \begin{cases} 0, & x \le 0 \\ 2, & 0 < x < 1 \\ 0, & x \ge 1 \end{cases}$, find $\int_{-3}^{3} f d\alpha$

Sol. As f(x) is continuous at 0 and also at 1, $\alpha(x)$ is a step function having jump point 0 and 1

$\int_{-3}^{3} f d\alpha = f(0)\left[\alpha\left(0^+\right) - \alpha\left(0^-\right)\right] + f(1)\left[\alpha\left(1^+\right) - \alpha\left(1^-\right)\right]$ = 3(2 – 0) + (-1) (0 – 2) = 8

Example 13 If $\alpha(x)=\begin{cases}2, & 0\le x\le 2\\ 5, & 2<x<3\\ 5, & x\ge 3\end{cases}$, find $\int_{-5}^{10} e^{-3x}\,d\alpha$

Sol. Let $f(x)=e^{-3x}$ is continuous function and $\alpha(x)$ is a step function, having jump points 0,2 and 3.

$$\int_{-5}^{10} e^{-3x}\,d\alpha = f(0)\left[\alpha(0^+)-\alpha(0^-)\right]+f(2)\left[\alpha(2^+)-\alpha(2^-)\right]+f(3)\left[\alpha(3^+)-\alpha(3^-)\right]$$

$$=1(2-0)+e^{-6}(5-2)+e^{-9}(6-5)=2+3e^{-6}+e^{-9}$$

Example 14 If f(x) = x and $\alpha(x)=x^2$ on [0, 1]. If $f\in R(\alpha)$. If yes, find value of $\int_0^1 f\,d\alpha$

Sol. As f(x) = x is continuous function on [0, 1]. Also $\alpha(x)=x^2$ is monotonic increasing function in [0, 1]. $\therefore\ f\in R(\alpha)$

Also $\int_0^1 f\,d\alpha=\int_0^1 x\,d\,x^2=\left[x^2.x\right]_0^1-\int_0^1 x^2\,dx=1-\frac{1}{3}=\frac{2}{3}$

Example 15 Evaluate the following R – S integrals

(i) $\int_0^2 x^2\,d\left(x^2\right)$ (ii) $\int_0^2 [x]\,d\left(x^2\right)$ (iii) $\int_0^6 \left(x^2+[x]\right)d\left(|3-x|\right)$

Sol. We know that $\int_a^b f\,d\alpha=\int_a^b f(x)\alpha'(x)\,dx$

(i) $\int_0^2 x^2 d\left(x^2\right) = \int_0^2 x^2 .2x dx = 2\int_0^2 x^3 dx = 2\left[\frac{x^4}{4}\right]_0^2 = 8$

(ii) $\int_0^2 [x] d\left(x^2\right) = \int_0^2 [x].2x dx = 2\int_0^1 [x].x dx + 2\int_1^2 [x].x dx$

$= 2\int_0^1 0 dx + 2\int_1^2 x dx = 2\left[\frac{x^2}{2}\right]_1^2 = 3$

(iii) $\int_0^6 \left(x^2 + [x]\right) d\left(|3 - \mathrm{x}|\right) = \int_0^3 \left(x^2 + [x]\right) d(3 - \mathrm{x}) + \int_3^6 \left(x^2 + [x]\right) d(\mathrm{x} - 3)$

$= -\int_0^3 \left(x^2 + [x]\right) d\mathrm{x} + \int_3^6 \left(x^2 + [x]\right) d\mathrm{x} = -\int_0^3 x^2 d\mathrm{x} - \int_0^3 [x] d\mathrm{x} + \int_3^6 x^2 d\mathrm{x} + \int_3^6 [x] d\mathrm{x}$

$= -\left[\frac{x^3}{3}\right]_0^3 - \int_0^1 [x] d\mathrm{x} - \int_1^2 [x] d\mathrm{x} - \int_2^3 [x] d\mathrm{x} + \left[\frac{x^3}{3}\right]_3^6 + \int_3^4 [x] d\mathrm{x} + \int_4^5 [x] d\mathrm{x} + \int_5^6 [x] d\mathrm{x}$

$= -\frac{1}{3}(27) - 0 - 1 - 2 + \frac{1}{3}(216 - 27) + 3 + 4 + 5$

$= -12 + 63 + 12 = 63$

Example 16. Give an example f a bounded function and α is increasing function on [a, b] such that $|f| \in R(\alpha)$ but $\int_a^b f d\alpha$ does not exist.

Sol. Let $\alpha(x)=|x|$ in [a, b] where a > 0 and $b<\infty$.

Let $f(x)=\begin{cases} 1, & x\in Q \\ -1, & x\in Q' \end{cases}$

T.P. $|f|\in R(\alpha)$ and $f\notin R(\alpha)$ on [a, b].

Since $\alpha(x)=|x|$ is monotonic increasing in [a, b]. Also $|f(x)|=1$

$\therefore |f|\in R(\alpha)$

T.P. $f\notin R(\alpha)$ on [a, b].

By definition $M_i=1, m_i=-1$, then

$U(P,f,\alpha)=\sum\limits_{i=1}^{n} M_i\Delta\alpha_i=\sum\limits_{i=1}^{n}\Delta\alpha_i=\alpha(b)-\alpha(a)=|b|-|a|=b-a$ and

$L(P,f,\alpha)=\sum\limits_{i=1}^{n} m_i\Delta\alpha_i=-\sum\limits_{i=1}^{n}\Delta\alpha_i=-\alpha(b)+\alpha(a)=|a|-|b|=a-b$ $\therefore$

$U(P,f,\alpha)\neq L(P,f,\alpha)$ $\therefore$ $f\notin R(\alpha)$ on [a, b].

Exercise

1. Prove that $\int_0^3 x\,d([x]) = \frac{3}{2}$.
2. Prove that $\int_0^3 x^2\,d([x]-x) = 5$
3. Let α be monotonic increasing in $[a,b]$, if $f \in R(\alpha)$ on $[a,b]$ then $f^2 \in R(\alpha)$ on $[a,b]$
4. Using the definition of RS – integral. Prove that $\int_a^{a+1} x^2\,d([x]) = (a+1)^2$
5. If f is continuous on $[0,n]$, where n is positive integer, then $\int_0^n f(x)\,d([x]) = f(1) + f(2) + \ldots.. + f(n)$
6. Let f be a bounded function defined on interval $[-1,1]$ and $\alpha : [-1,1] \to R$ be defined by $\alpha(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ 1, & x > 0 \end{cases}$. Show that $f \in R(\alpha)$ if and only if f is continuous at x = 0 and then $\int_{-1}^{1} f\,d\alpha = f(0)$

The book is intended to serve as a text in analysis by the Graduate and Post – Graduate students of the various universities.

Key features include:

- ★ A broad view of mathematics throughout the book.
- ★ Elegant proofs.
- ★ Excellent choice of topics.
- ★ Numerous examples and exercises to reinforce methodology.

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